Es. Complesse (MOLTENI '15/'16)

[1] Rec. 2. Rosso

→ $\mathbb{C}$ varie strutture:

→ $\mathbb{R}$-spazio Vettoriale $(x,y) \leftrightarrow x+iy$ con x e $y \in \mathbb{R}$.

→ MA è anche algebra, rispetto al prodotto

$$(x_1,y_1)(x_2,y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1)$$

(ovvero $(x_1+iy_1)(x_2+iy_2) = \dots$ con $i^2 = -1$.

→ Prodotto commutativo, c'è unità $1 = (1,0)$ e $0 = (0,0)$.

→ Se $(x,y) \neq (0,0)$ allora

$$(x,y)\cdot\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = (1,0)$$

quindi $\mathbb{C}$ è Campo.

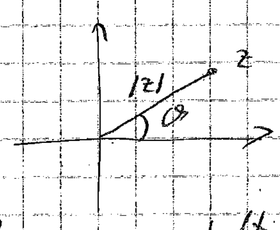
→ Il Campo Non è ordinato: nei campi ordinati $x^2 \geq 0$, mentre $i^2 = -1 < 0$

→ $\mathbb{C}$ è campo algebricamente chiuso ovvero se $P(x) \in \mathbb{C}[x]$ allora $P(x) = 0$ ha esattamente ∂P soluzioni (contate con molteplicità).

Varie dim. nel corso se ne vedranno un paio (credo).

→ se $z \in \mathbb{C}^*$ (ovvero $z \neq 0$) allora $z = |z| \cdot e^{i\theta_z}$

con $|z| \in \mathbb{R}_{>0}$ e θ_z ANGOLO $\in \mathbb{R}/_{2\pi\mathbb{Z}}$



Utile per prodotti, perché $z = |z| e^{i\theta_z}$ e $w = |w| e^{i\theta_w}$

Allora $z \cdot w = |z||w| e^{i(\theta_z + \theta_w)}$ quindi $|zw| = |z||w|$ e

$\theta_{zw} = \theta_z + \theta_w$ Modulo 2π.

Questo perché $e^{i\theta} = \cos\theta + i\sin\theta$ quindi

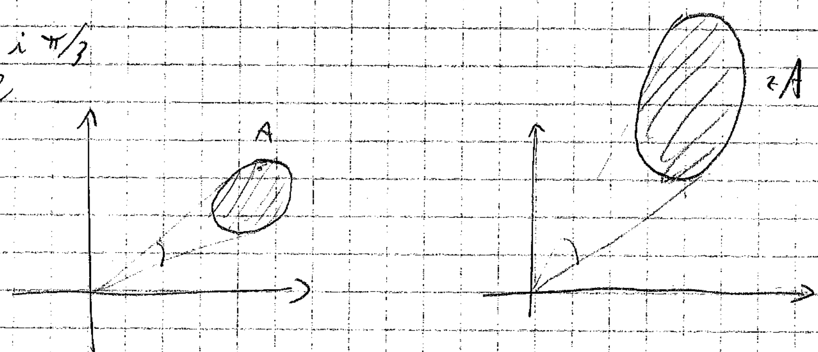
$$e^{i\theta_z}\cdot e^{i\theta_w} = (\cos\theta_z + i\sin\theta_z)(\cos\theta_w + i\sin\theta_w)$$
$$= \cos\theta_z\cos\theta_w - \sin\theta_z\sin\theta_w + i(\cos\theta_z\sin\theta_w + \sin\theta_z\cos\theta_w)$$
$$= \cos(\theta_z+\theta_w) + i\sin(\theta_z+\theta_w)$$
$$= e^{i(\theta_z+\theta_w)}$$

Quindi, se $z\neq 0$, $x \longrightarrow z\cdot x$ è $= |z|e^{i\theta_z}x$

dilatazione fattore $|z|$

rotazione angolo θ_z

$z = 2e^{i\pi/3}$



Dilatato fattore 2, ruotato $\frac{\pi}{3}$

→ $\frac{1}{-3+2i} = \frac{1}{-3+2i}\cdot\frac{-3-2i}{-3-2i} = \frac{-3-2i}{9+4} = -\frac{3}{13} - \frac{2}{13}i$

→ $(z-2i)^4 = (3+5i)^4$

$z-2i = (3+5i)e^{\frac{2\pi}{4}j}$ $\quad j=0,1,2,3$

$\Rightarrow z = 2i + (3+5i)e^{\frac{j\pi}{4}}$ $\quad$ "

→ Distinguere tra le radici k-esime (un insieme di k numeri complessi, eventualmente ripetuti) e la funzione $\sqrt[k]{z}$ che definiremo (ove non lo è se non per $z\in\mathbb{R}_{>0}$).

→ $\mathbb{C}$ con $|\,|$ è spazio normato (poiché coincide con $(\mathbb{R}^2, |\,|_{eucl})$).

OSS In spazi normati finiti-dim tutte le norme sono equivalenti, ovvero date $\|\,\|_1$ e $\|\,\|_2$ esistono sempre le costanti c_1 e $c_2>0$ t.c.

$$c_1\|\,\|_1 \le \|\,\|_2 \le c_2\|\,\|_1$$

Ad esempio $\frac{1}{\sqrt{2}}(|a|+|b|) \le \sqrt{a^2+b^2} \le |a|+|b|$

così $(\mathbb{C}, |\,|)$ equivale alla convergenza in $|\,|$ assoluto sulle coordinate. ovvero

$$z_n \to z \quad \text{in } \mathbb{C} \text{ } |\cdot| \iff \begin{cases} \mathrm{Re}\, z_n \to \mathrm{Re}\, z & \text{in } \mathbb{R} \\ \mathrm{Im}\, z_n \to \mathrm{Im}\, z & \text{in } \mathbb{R} \end{cases}$$

In particolare $(\mathbb{C}, |\cdot|)$ è di BANACH (perché $\mathbb{R}$ con $|\,|$ lo è).

(-) In particolare $\sum_{k=1}^{\infty} z_k$ converge in $\mathbb{C}$

$\iff S_n := \sum_{k=1}^{n} z_k$ è di CAUCHY

$\implies z_n = S_n - S_{n-1} \to 0$ (C. n.)

altrimenti non sufficiente (non lo è in $\mathbb{R}$, e $\mathbb{R} \subseteq \mathbb{C}$).

(-) Σ di potenze:

$\sum_{n=0}^{\infty} a_n (z-z_0)^n$ $\quad a_n \in \mathbb{C}$, $\quad z_0 \in \mathbb{C}$ è "centro",

Normalizzo: $z_0 = 0$

Abel se $\sum_{n=0}^{\infty} a_n z^n$ converge in ξ, allora converge in Disco $(z : |z| < |\xi|)$. Conv. Assoluta e Unif in compatti.

Prof siccome $a_n \xi^n \to 0$ quindi $|a_n \xi|^n$ limitato, da L.

allora $|a_n z^n| = |a_n \xi^n| \cdot \left|\frac{z}{\xi}\right|^n$ (se $\xi \ne 0$, ovvio).

$\le L \cdot \left|\frac{z}{\xi}\right|^n \leftarrow |\,| < 1$ quindi converge per confronto (Weierstrass) ▨

Def $\rho := \mathrm{Sup}\,\{|\xi|,$ dove $\sum a_n \xi^n$ converge$\}$.

allora la Σ converge in D_ρ e non converge in $\mathbb{C} \setminus \overline{D_\rho} \leftarrow$ chiusura

Calcolo: $\rho = 1/l$ (con $1/l = \begin{cases} 0 & l = \infty \\ \infty & l = 0 \end{cases}$)

dove $l = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

oppure: se $\exists\ l = \lim_{n\to\infty} |a_{n+1}|/|a_n|$, $\rho = 1/l$.

__Esempio__ $\sum_{n=1}^{\infty} \frac{z^n}{n} \rightarrow \rho = 1$

Converge in $|z| < 1$. __Non__ converge in $|z| > 1$.

Sul __bordo__?

Verificare che converge in $|z| = 1$, $z \neq 1$ (somme parziali).

__Esempio__ $\sum_{n=0}^{\infty} \left(\frac{n-5i}{in+3}\right)^{n^4} z^{n^2}$. $\rho = ?$

$l = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

ma $\sqrt[n]{|a_n|} = \begin{cases} 0 & \text{se } n \text{ \underline{Non} è quadrato} \\ \left(\left|\frac{k-5i}{ik+3}\right|^{k^4}\right)^{\frac{1}{k^2}} & \text{se } n = k^2 \end{cases}$

$$\left(\left|\frac{k-5i}{ik+3}\right|^{k^4}\right)^{1/k^2} = \exp\left(k^2 \log\left(\left|\frac{k-5i}{ik+3}\right|\right)\right) = \exp\left(k^2 \log\left(\left|\frac{1-\frac{5i}{k}}{1-3i/k}\right|\right)\right)$$

$$= \exp\left(\frac{k^2}{2} \log\left(\frac{(1+\frac{5^2}{k^2})}{1+\frac{9}{k^2}}\right)\right) = \exp\left(\frac{k^2}{2} \cdot \frac{16+o(1)}{k^2}\right)$$

$$\rightarrow \exp(8)$$

$\underline{\underline{\rho = e^{-8}}}$.

2 La mappa $z \to 1/\bar{z}$ inversione circolare

Siano $A, B, \alpha, \beta \in \mathbb{R}$. le rette e circ. del piano hanno equazione

Reali: | complessi

$A(x^2+y^2)+\alpha x+\beta y+B=0$ | $2A|z|^2+\bar{z}_0 z+z_0\bar{z}+2B=0$ (*)

$z=x+iy$ e $z_0=\alpha+i\beta$

la mappa $z \to w=1/\bar{z}$ trasforma (*) in:

$$2A\frac{1}{|w|^2}+\bar{z}_0\frac{1}{\bar{w}}+z\frac{1}{w}+2B=0 \iff 2B|w|^2+\bar{z}_0 w+z_0\bar{w}+2A=0$$

quindi manda {circ. e rette} $\Rightarrow$ {circ. e rette}

oss 1 caso $A=0, B\neq 0$ è retta $\Rightarrow$ circ.

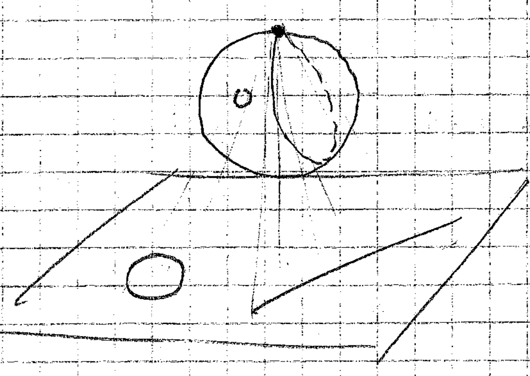
$A\neq 0, B\neq 0$ è circ. $\Rightarrow$ circ.

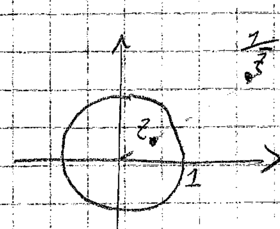
$A=0, B=0$ è retta $\Rightarrow$ retta

$A\neq 0, B=0$ è circ. $\Rightarrow$ retta.

oss 2 la mappa $z \to 1/\bar{z}$ non è definita per $z=0$ quindi in realtà alcune circ. e rette sono prive di 1 punto.

Questo perché la mappa andrebbe definita su $\bar{\mathbb{C}}$ o sfera di Riemann in cui, tra l'altro, non c'è più distinzione tra circ e retta ma solo circonferenze.





oss 3 Inversione circolare:

$$z=\rho e^{i\theta} \to \frac{1}{\bar{z}}=\frac{1}{\rho}e^{i\theta}$$

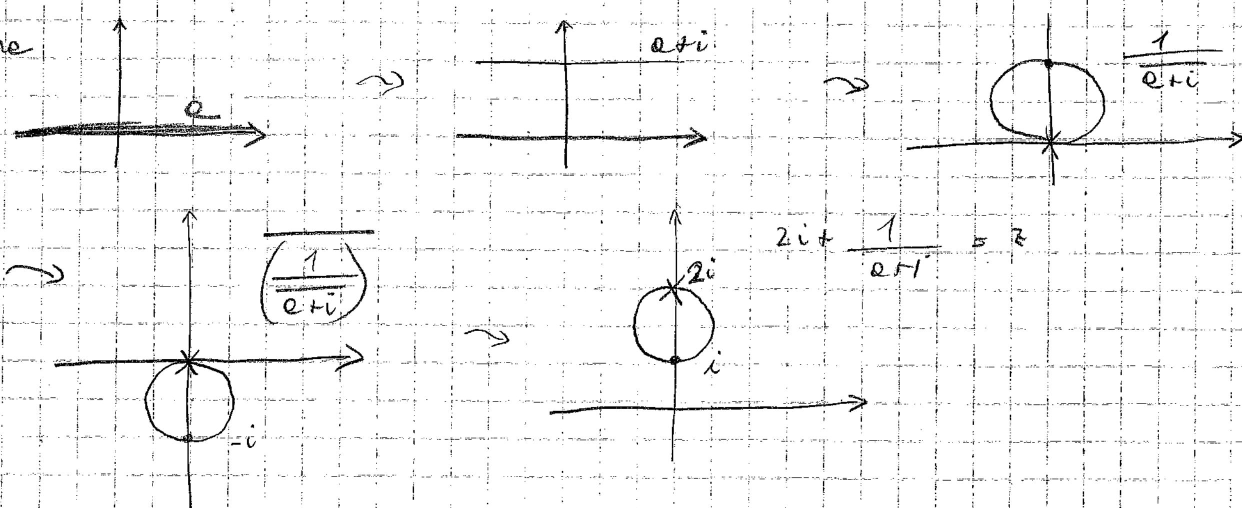
ovvio: "prodotto distanze" $=1$ e angolo uguale (questa è la def geometrica solita).

es) Descrivere $B = \{ z \in \mathbb{C} : \frac{z-i}{2+iz} \in \mathbb{R} \}$ es. 1.5. foglio 1 6

Allora $\frac{z-i}{2+iz} = a \Rightarrow z - i = 2a + iaz$

$$\Rightarrow z = \frac{i+2a}{1-ia} = \frac{2ia-1}{a+i} = 2i + \frac{1}{a+i} = 2i + \overline{\left(\frac{1}{a+i}\right)}$$

Allora



es. 2.7] foglio 2

Verificare $|e^z - 1| \le e^{|z|} - 1 \le |z| e^{|z|}$ e che $|z| \le 1 \Rightarrow |e^z - 1| \le 2|z|$

Proof

① $|e^z - 1| = \left| \sum_{n=1}^{\infty} \frac{z^n}{n!} \right| \le \sum_{n=1}^{\infty} \frac{|z|^n}{n!} = e^{|z|} - 1$

inoltre $e^{|z|} - 1 = \sum_{n=1}^{\infty} \frac{|z|^n}{n!} = |z| \sum_{n=1}^{\infty} \frac{|z|^{n-1}}{n!} \le |z| \sum_{n=1}^{\infty} \frac{|z|^{n-1}}{(n-1)!} = \underline{\underline{|z| e^{|z|}}}$

② $|e^z - 1| \le \sum_{n=1}^{\infty} \frac{|z|^n}{n!} = |z| \sum_{n=1}^{\infty} \frac{|z|^{n-1}}{n!}$ se $|z| \le 1$ $\le |z| \sum_{n=1}^{\infty} \frac{1}{n!} = (e-1)|z| \le 2|z|$

es. 2.8] foglio 2:

$|\cos(iz) + \sinh(z)| = e^{iz}$

$\cos(iz) + \sinh z = \frac{e^{i(iz)} + e^{-i(iz)}}{2} + \frac{e^{z} - e^{-z}}{2} = e^z$ quindi

$|e^z| = e^{iz}$ se $z = x+iy$ $(x, y \in \mathbb{R})$ $\Leftrightarrow e^x = e^{ix-y} \Leftrightarrow e^{x+y} = e^{ix}$

quindi $e^{ix} \in \mathbb{R}$ $(\Rightarrow x = k\pi)$ e > 0 $(\Rightarrow x = 2k\pi)$ $\Rightarrow e^{x+y} = 1 \Rightarrow y = -x$

$\Rightarrow \underline{\underline{z = 2k\pi(1-i)}}$; $k \in \mathbb{Z}$.

Le potenze

è stato def. ARG(z), come

$$\mathrm{Arg} : S := \mathbb{C} \setminus \{(-\infty, 0]\} \longrightarrow (-\pi, \pi)$$

e $\mathrm{Log} : S \to \mathbb{C}$ ponendo

$$\mathrm{Log}(z) := \log|z| + i\,\mathrm{Arg}(z)$$

Sappiamo che questo può dare problemi: In genere non è vero che

$$\mathrm{Log}(\alpha\beta) = \mathrm{Log}(\alpha) + \mathrm{Log}(\beta)$$

(Banalmente: $\mathrm{Log}(e^{i\frac{2}{3}\pi} \cdot e^{i\frac{2}{3}\pi}) = \mathrm{Log}(e^{i\frac{4}{3}\pi}) = \mathrm{Log}(e^{-i\frac{2}{3}\pi}) = \underline{\underline{-i\frac{2}{3}\pi}}$

Mentre $\mathrm{Log}(e^{i\frac{2}{3}\pi}) + \mathrm{Log}(e^{i\frac{2}{3}\pi}) = \underline{\underline{i\frac{2}{3}\pi + i\frac{2}{3}\pi}}$)

Dovuto al fatto che Arg non è additiva (ma lo è modulo 2π ...)

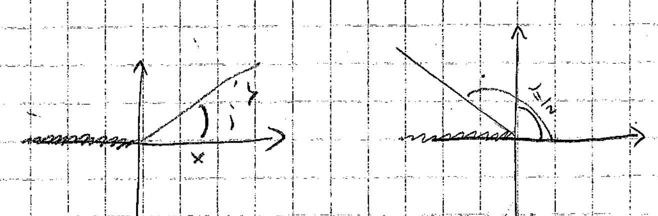
Log è olomorfa in S, con $\mathrm{Log}' z = \dfrac{1}{z}$

Dim Log è $C^1(S) \to \mathbb{R}^2$ quindi basta verificare che valgono le Cauchy-Riemann. Nelle note si usano le C-R polari.

E le cartesiane?

$$u = (\mathrm{Re}\,\mathrm{Log})(x+iy) := \log(\sqrt{x^2+y^2}) = \frac{1}{2}\log(x^2+y^2)$$

$$v = (\mathrm{Im}\,\mathrm{Log})(x+iy) = \mathrm{Arg}(x+iy) = \begin{cases} x>0 & \mathrm{arctg}\left(\frac{y}{x}\right) \\ y>0 & \frac{\pi}{2} - \mathrm{arctg}\left(\frac{x}{y}\right) \\ y<0 & -\frac{\pi}{2} - \mathrm{arctg}\left(\frac{x}{y}\right) \end{cases}$$



ora, $\partial_x v = \quad x>0 \;\; \dfrac{-y}{x^2+y^2} \;;\quad y>0 \;\; \dfrac{-y}{x^2+y^2} \;;\quad y<0 \;\; \dfrac{-y}{x^2+y^2}$

ovvero $\partial_x v = \dfrac{-y}{x^2+y^2}$ in S. ANALOGO: $\partial_y v = \dfrac{x}{x^2+y^2}$.

Inoltre $\partial_x u = \dfrac{x}{x^2+y^2}$ e $\partial_y u = \dfrac{y}{x^2+y^2}$ $\Rightarrow$ $\partial_x u = \partial_y v$ vero e

$\partial_y u = -\partial_x v$ vero $\Rightarrow$ OK.

e $Lg'(z) = \partial_x U + i\partial_x V = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{\bar z}{|z|^2} = \frac{1}{z}$ ▨

8

Potenze

$$\{\alpha^\beta\} := \{e^{\beta\, lg\,\alpha}\}$$ come insieme di numeri

quindi $\{(-1)^i\} = \{e^{i\, lg(-1)}\} = \{e^{i(i\pi+2k\pi i)}\}_{k\in\mathbb{Z}}$

$$= \{e^{-\pi+2k\pi}\}_{k\in\mathbb{Z}}$$

Ma come fissare?

Se $\alpha\in\mathbb{C}$, $\quad z^\alpha := \exp(\alpha\, Lg\, z)$ per $z\in S = \mathbb{C}\setminus\{(-\infty,0]\}$

Allora $z^\alpha\cdot z^\beta = \exp(\alpha Lg z)\exp(\beta Lg z) = \exp((\alpha+\beta)Lg z) = z^{\alpha+\beta}$

MA $\quad z^\alpha\cdot w^\alpha \overset{?}{=} (zw)^\alpha \quad$ problema legato a Lg.

esempio $\alpha = i$, $z = w = e^{\frac{2}{3}\pi i}$ allora

$z^\alpha\cdot z^\alpha = \exp(i\cdot Lg(e^{\frac{2}{3}\pi i}))\cdot\exp(i\, Lg(e^{\frac{2}{3}\pi i})) = \underline{\underline{\exp(-\frac{4}{3}\pi)}}$

e $(z\cdot z)^\alpha = \exp(i\cdot Lg(e^{\frac{4}{3}\pi i})) = \exp(i\cdot i(-\frac{2}{3}\pi)) = \underline{\underline{\exp(\frac{2}{3}\pi)}}$

le potenze $(1+z)^{\alpha}$.

→ Sia $\alpha \in \mathbb{C}$, fissato.

Sia $\binom{\alpha}{n} = \begin{cases} 1 & \text{se } n=0 \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} & n \geq 1. \end{cases}$ Binomiali generalizzati.

$$\frac{\binom{\alpha}{n+1}}{\binom{\alpha}{n}} = \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!} \cdot \frac{n!}{\alpha(\alpha-1)\cdots(\alpha-n+1)} = \frac{\alpha-n}{n} \longrightarrow 1$$

quindi $S_{\alpha}(z) = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^{n}$ ha raggio 1 — ERRORE!

Solo se ha senso dividere, ⇒ solo se $\alpha \notin \mathbb{N}$. (nel caso $\alpha \in \mathbb{N}$ è Polinomio ⇒ è raggio ∞).

Chi è $S_{\alpha}(z)$? Oss. $S_{\alpha}(0) = 1$.

Inoltre

$$(1+z)\, S_{\alpha}'(z) = (1+z) \sum_{n=0}^{\infty} \binom{\alpha}{n} n z^{n-1} = \sum_{n=0}^{\infty} \left(\binom{\alpha}{n+1}(n+1) + \binom{\alpha}{n} n \right) z^{n}$$

ma $\binom{\alpha}{n+1}(n+1) + \binom{\alpha}{n} n = \alpha$ se $n=0$

e se $n \geq 1$ $\quad \dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)(\alpha-n)}{n!} + \dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \cdot n$

$$= \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \left[(\alpha-n) + n \right] = \alpha \binom{\alpha}{n}$$

quindi $\begin{cases} (1+z)\, S_{\alpha}'(z) = \alpha\, S_{\alpha}(z) \\ S_{\alpha}(0) = 0 \end{cases} \Rightarrow S_{\alpha}(z) = \underline{\underline{(1+z)^{\alpha}}}$ almeno se z è <u>reale</u>.

<u>Ma</u> $S_{\alpha}(z)$ è <u>olomorfa</u> in $D_0(1)$ (in $\mathbb{C}$ se α intero ≥ 0) quindi per l'unicità del prolungamento <u>deve</u> coincidere con

$\exp(\alpha \operatorname{Log}(1+z))$ dove $\operatorname{Log}(1+z)$ la <u>determinazione</u> del log complesso che è reale per <u>z reale > -1</u>.

Conseguenza "Combinatorica") $(1+z)^{\alpha}(1+z)^{\beta} = (1+z)^{\alpha+\beta}$

$$\Rightarrow \binom{\alpha+\beta}{n} = \sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta}{n-k} \qquad \forall \alpha,\beta \in \mathbb{C},\ \forall n \in \mathbb{N}$$

(difficile da dimostrare direttamente)

oss Analitica $\sqrt{1+z}\cdot\sqrt{1+z} = 1+z$

quindi "raggio 1" · "raggio 1" ha raggio ∞ in questo caso

In generale, è noto che $\sum a_n$ converge e $\sum b_n$ converge ASS, allora
$\sum_n \left(\sum_{k=0}^{n} a_k b_{n-k}\right)$ converge a $(\sum a_n)\cdot(\sum b_n)$.

In particolare, se F e G sono $\sum$ di potenze con raggio ρ_F e ρ_G, allora
$F\cdot G$ è $\sum$ di potenze con $\rho_{FG} \geq \min(\rho_F, \rho_G)$.
L'esempio precedente mostra caso di $>$ stretto.

Def. Un ramo olomorfo di $(1-z^2)^{\frac12}$ per $|z|>1$.

ovvero $(1-z^2)^{\frac12} = (-z^2)^{\frac12}\left(1-\frac{1}{z^2}\right)^{\frac12}$

scelgo iz

quindi un ramo olomorfo è $iz\left(1-\frac{1}{z^2}\right)^{\frac12}$ ☒

Merc. 16. 3. '16

Oss. $f: \Omega \subset \mathbb{C} \to \mathbb{C}$. $f(z) = A(z) + i\,B(z)$ con $A = \mathrm{Re}\, f$, $B = \mathrm{Im}\, f$.

Allora $f(z)\,dz = (A + iB)(dx + i\,dy)$

$$= \underbrace{(A\,dx - B\,dy)} + i\underbrace{(A\,dy + B\,dx)}$$

forme diff.

f soddisfa C-R $\iff \begin{cases} \partial_x A = \partial_y B \\ \partial_y A = -\partial_x B \end{cases} \iff$ le due forme sono chiuse

(e quindi saranno esatte sui sempl. connessi).

Inoltre, se f è C-R ($\Rightarrow$ olomorfa) con primitiva F, allora $\mathrm{Re}\, F$ è un potenziale per $A\,dx - B\,dy$ e $\mathrm{Im}\, F$ è un potenziale per $A\,dy + B\,dx$.

Es. $\displaystyle\int_\gamma (\bar{z} + (z-1)^8)\, dz$ $\qquad \gamma: 0 \xrightarrow{t} 1 \quad t + it^2$

$$\int_\gamma (\bar{z} + (z-1)^8)\,dz = \int_\gamma \bar{z}\, dz + \frac{1}{9}(z-1)^9\Big|_{\gamma(0)}^{\gamma(1)} = \int_\gamma \bar{z}\,dz + \frac{1}{9}(z-1)^9\Big|_0^{1+i}$$

$$= \int_\gamma \bar{z}\,dz + \frac{i^9 - (-1)^9}{9} = \int_\gamma \bar{z}\,dz + \frac{i+1}{9}$$

$$= \int_\gamma (x - iy)(dx + i\,dy) + \frac{i+1}{9} = \int_\gamma \underbrace{(x\,dx + y\,dy)}_{d\left(\frac{x^2+y^2}{2}\right)} + i\int_\gamma (x\,dy - y\,dx) + \frac{i+1}{9}$$

$$= \frac{1}{2}(x^2+y^2)\Big|_0^{1+i} + i\int_0^1 (t\, d(t^2) - t^2\,dt) + \frac{i+1}{9}$$

$$= i\int_0^1 (2t - t^2)\,dt + 1 + \frac{i+1}{9} = i\left[t^2 - \frac{t^3}{3}\right]_0^1 + 1 + \frac{i+1}{9} = \frac{2}{3}i + 1 + \frac{i+1}{9} = \frac{10+7i}{9}$$

OSS $Re(z^n)$ e $Im(z^n)$ nelle variabili x e y ($= Re z$ e $Im z$) (12)

Sono Polinomi omogenei di grado n.

-) Sono indip. (su $\mathbb{C}$) $\overset{n\neq 0}{\text{}}$ perché $\alpha Re(z^n) + \beta Im(z^n) =$

$$= \alpha \frac{z^n + \bar z^n}{2} + \beta \frac{z^n - \bar z^n}{2i} = 0 \quad \forall z \Rightarrow$$

$$(\alpha - i\beta) z^n = (-\alpha - i\beta) \bar z^n$$

Se $\alpha = i\beta$ allora $-2i\beta \bar z^n = 0 \ \forall z \Rightarrow \beta = 0 \Rightarrow \alpha = 0.$

se $\alpha \neq i\beta$ preso $z \neq 0$ $\left(\frac{z}{\bar z}\right)^n = -\frac{\alpha + i\beta}{\alpha - i\beta}$ imp. (SX ha ∞ valori $(n\neq 0)$, DX uno solo).

-) $$\Delta(Re(z^n)) = \frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} Re(z^n) = 0$$

$$\Delta(Im(z^n)) = \frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} Im(z^n) = 0$$

$\Rightarrow$ $Re(z^n)$ e $Im(z^n)$ sono Armonici.

$n = 1$: x, y $\qquad$ $n=2$: $x^2 - y^2$, $2xy$ $\qquad$ $n=3$: $x^3 - 3xy^2$, $-3x^2y + y^3$

-) Verificare che l'{Polinomi di x, y, omogenei di grado n ha dim $\underline{\underline{2}}$}

(se $n \neq 0$) $\Rightarrow$ {Pol. omo grado n} $= \langle Re(z^n), Im(z^n)\rangle$.

Dim lo Spazio è Vettoriale (facile)

$x^n, x^{n-1}y, x^{n-2}y^2, \dots, xy^{n-1}, y^n$ è Base

$$P(x,y) = \sum_{j=0}^{n} \alpha_j x^j y^{n-j}$$

$$0 = \Delta P = \sum_{j=0}^{n} \alpha_j j(j-1) x^{j-2} y^{n-j} + \sum_{j=0}^{n} \alpha_j (n-j)(n-j-1) x^j y^{n-2-j}$$

$$= \sum_{j=0}^{n-2} \underbrace{[\alpha_{j+2}(j+2)(j+1) + \alpha_j (n-j)(n-j-1)]}_{} x^j y^{n-2-j}$$

$\Rightarrow$ Ricorsione ordine 2 $\Rightarrow$ al + due Soluzioni.

OSS se $S: \mathbb{C}[x,y] \to \mathbb{C}[x,y]$ $(SP)(x,y) = P(-x,y)$. Allora $S\Delta = \Delta S \Rightarrow$ esiste base di {Pol. Armonici grado n} $\setminus \{0\}$ Autovettori di S. Infatti $S(Re f(z)) = (-1)^n Re f(z)$, $S(Im f(z)) = -(-1)^n Im f(z)$ se $n \geq 2$.

Calcolare $\int_\gamma (\operatorname{Ch} z - \operatorname{ch}(\bar z))\, dz \qquad \gamma : 0 \xrightarrow{t} 1 \qquad (1+i)t$

$$= \int_\gamma \operatorname{ch} z\, dz - \int_\gamma \operatorname{ch}(\bar z)\, dz = \operatorname{Sh} z \Big|_0^{1+i} - \int_\gamma \operatorname{ch}(x-iy)\, dz$$

$$= \operatorname{Sh}(1+i) - (1+i)\int_0^1 \operatorname{ch}(t-it)\, dt$$

$$= \operatorname{Sh}(1+i) - (1+i)\int_0^1 [\operatorname{ch}(t)\operatorname{ch}(-it) + \operatorname{Sh}(t)\operatorname{Sh}(-it)]\, dt$$

$$= \operatorname{Sh}(1+i) - (1+i)\int_0^1 [\operatorname{ch} t \cos t - i \operatorname{Sh} t \sin t]\, dt$$

oppure

$$= \operatorname{Sh}(1+i) - (1+i) \frac{\operatorname{Sh}((1-i)t)}{1-i}\Big|_0^1$$

$$= \operatorname{Sh}(1+i) - (1+i) \cdot \frac{\operatorname{Sh}(1-i)}{1-i}$$

$$= \frac{(1-i)\operatorname{Sh}(1+i) - (1+i)\operatorname{Sh}(1-i)}{1-i} = \frac{2i}{1-i} \operatorname{Im}\left((1-i)\operatorname{Sh}(1+i)\right)$$

$$= \frac{2i}{1-i} \operatorname{Im}\left((1-i)(\operatorname{Sh} 1 \cosh i + \operatorname{Ch} 1 \sinh i)\right)$$

$$= \frac{2i}{1-i} \operatorname{Im}\left((1-i)(\operatorname{Sh} 1 \cos 1 + i \operatorname{Ch} 1 \sin 1)\right)$$

$$= \frac{2i}{1-i} (\operatorname{Ch} 1 \sin 1 - \operatorname{Sh} 1 \cos 1)$$

$$= (-1+i)(\operatorname{ch} 1 \sin 1 - \operatorname{Sh} 1 \cos 1)$$

---

oppure $\int_\gamma \operatorname{ch}(\bar z)\, dz$

-) Det ramo di $\sqrt{1-z^2}$ con $|z|>1$.

Modo facile: $1-z^2 = -z^2\left(1-\frac{1}{z^2}\right)$ e $1-\frac{1}{z^2}$ è holo per $|z|>1$.

quindi $S_{\frac{1}{2}}\left(-\frac{1}{z^2}\right) = \sum_{n=0}^{\infty} \binom{1/2}{n} \left(-1/z^2\right)^n$ è holo per $|z|>1$ e

Sappiamo che $\left(S_{\frac{1}{2}}(w)\right)^2 = 1+w^2$.

quindi ramo olomorfo in $|z|>1$ è:

$$\underline{\underline{i z S_{1/2}\left(-\frac{1}{z^2}\right)}}$$

Metodo complicato: Trovare $f(z)$ olomorfa in D t.c. $(P(z))^{\frac{1}{m}}$ con $P(z) \in \mathbb{C}[z]$, $m \in \mathbb{N}$, $m>0$.

In realtà si chiede di trovare ramo olomorfo in D di

$$(*) \qquad y(z)^m = P(z)$$

oss: $P(z) = c \prod_{n=1}^{d} (z-\alpha_n)^{e_n}$ $\quad \partial P = \sum_{n=1}^{d} e_n$ $\quad \{\alpha_n\}_1^d$ radici distinte

$c \neq 0$.

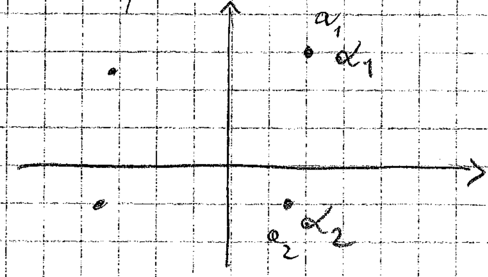
da $(*)$ $\quad m\frac{y'}{y} = \sum_{n=1}^{d} \frac{e_n}{z-\alpha_n}$ $\quad$ (Derivata logaritmica)

la forma è chiusa in $\mathbb{C} \setminus \{\alpha_n\}_{n=1}^{d}$

$\Rightarrow$

$$y(z) = c_m \exp\left[\frac{1}{m} \int_{z_0 \gamma z} \sum_{n=1}^{d} \frac{e_n}{w-\alpha_n}\, dw\right]$$

c_m una delle radici m-esime di c.

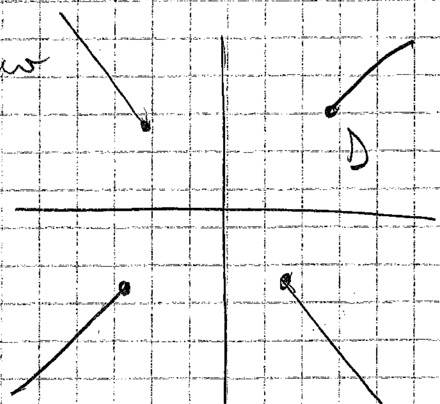


quindi se D ${}^{\vee}$ è (Aperto, connesso) sempl connesso in $\mathbb{C} \setminus \{\alpha_n\}_{n=1}^{d}$ 15

c'è Potenziale ($\Rightarrow$ primitiva) quindi Prendo z_0 in D,

z in D e γ qualunque cammino $z_0 \to z$.

Esempio



D è ok.

MA ogni termine $\frac{e_n}{w-\alpha_n}$ se integrato in $\odot_{\alpha_n}$ dà

$2\pi i\, e_n$ quindi dato D (aperto connesso) è t.c.

$\forall\ \gamma$ chiuso in D si ha

$$\int_\gamma \sum_n \frac{e_n}{w-\alpha_n}\, dw = 2\pi i \sum_{n=1}^{d} e_n\, \delta_\gamma(\alpha_n) \quad \text{con } \delta_\gamma(\alpha_n) = \begin{cases} 1 & \text{se } \gamma \text{ circonda } \alpha_n \\ 0 & \text{altrimenti.} \end{cases}$$

Quindi se D è t.c. $m \mid \sum_{n=1}^{d} e_n \delta_\gamma(\alpha_n)$ $\forall$ curva chiusa γ

in D, allora

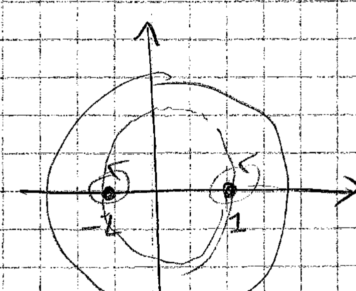
$$c_m \cdot \exp\left[\frac{1}{m} \int_{z_0 \gamma z} \sum_{n=1}^{d} \frac{e_n}{w-\alpha_n}\, dw \right]$$

è ben definito

e in D dà rami olomorfi.

Esempio $\sqrt{1-z^2}$ in $|z|>1$

due radici di mult. 1 $\Rightarrow$



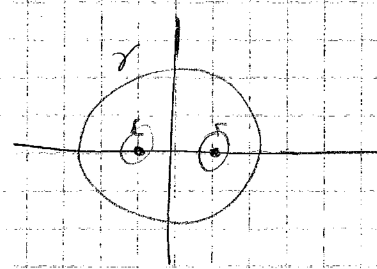
se $D = \{|z|>1\}$ ogni cammino in D

dà $2 \mid \sum_{n=1}^{2} \delta_\gamma(\alpha_n) \Rightarrow$ esiste rami olomorfi.

se però si cerca $(1-z^2)^{\frac{1}{3}}$ olomorfa in

$D = \{|z| > 1\}$ Non c'è!

Perché $3 \nmid \sum_{n=1}^{2} \delta_\gamma(\alpha_n) = 2 \Longrightarrow$

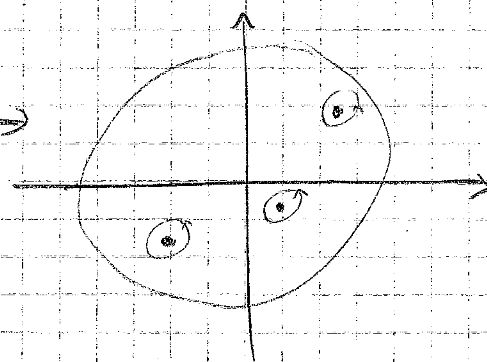


Se però $P(z)$ ha 3 radici distinte

allora $(P(z))^{\frac{1}{3}}$ ha olomorfa in $D = \{|z| > c\}$

quando c grande abbastanza:

Perché allora $3 \mid \sum_{n=1}^{3} \delta_\gamma(\alpha_n) = 3 \Longrightarrow$



•) Σ di potenze e f. generatrici.

supp. di avere data:

$$\begin{cases} a_{n+k} + \alpha_1 a_{n+k-1} + \alpha_2 a_{n+k-2} \cdots + \alpha_k a_n = \beta_n & (*) \\ a_0 = c_0 \\ a_1 = c_1 \\ \cdots \\ a_{k-1} = c_{k-1} \end{cases}$$

con $\alpha_1, \dots, \alpha_k \in \mathbb{C}$, $\{\beta_n\}_{n=0}^{\infty}$ e $c_0, \dots, c_{k-1} \in \mathbb{C}$ assegnati

le $\{a_n\}_{n=0}^{\infty}$ è univ. determinata. Vogliamo studiare l'andamento di a_n.

Se $F(z) := \sum_{n=0}^{\infty} a_n z^n$ e $G(z) = \sum_{n=0}^{\infty} \beta_n z^n$

Moltiplichiamo (*) per z^{n+k} e sommiamo su $\underline{n}$. si ha

$$\sum_{n=0}^{\infty} a_{n+k} z^{n+k} = \sum_{n=k}^{\infty} a_n z^n = F(z) - (c_0 + c_1 z + \cdots + c_{k-1} z^{k-1})$$

$$\sum_{n=0}^{\infty} a_{n+k-1} z^{n+k} = z \sum_{n=k-1}^{\infty} a_n z^n = z F(z) - z(c_0 + c_1 z + \cdots + c_{k-2} z^{k-2})$$

!

Quindi (*) diventa:

$$P(z) F(z) - Q_c(z) = z^k G(z)$$

dove $Q_c(z)$ è $\underline{\text{pol.}}$ dipendente dalle $\underline{c}$ e grado $\leq k-1$, mentre $P(z)$

è Pol. $1 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_k z^k$.

Quindi $F(z) = \dfrac{z^k G(z) + Q_c(z)}{P(z)}$

che è sol. esplicita. Da qui si possono ricavare le a_n esplicitamente,

$\underline{\text{Ma}}$ per avere l'ordine di grandezza di $|a_n|$ possiamo procedere + facilmente così:

Sia $P(z) = c \prod_{j=1}^{\partial P} (z - \gamma_j)$ la fattorizzazione di P in $\mathbb{C}$.

con $|\gamma_1| \leq |\gamma_2| \leq |\gamma_3| \cdots$

Supp. G abbia raggio di convergenza $> |\gamma_{\partial P}|$ così che è $\underline{\text{olomorfa}}$

nel disco aperto contenente le γ_j.

Visto λ_1 è zero del denominatore, a meno che lì s'annulli anche il numeratore questo sarà singolarità di F, e visto che il raggio di F è = distanza dalla 1ª singolarità si ha

$$\overline{\lim_{n\to\infty}} |a_n|^{\frac{1}{n}} = \frac{1}{\rho} = \frac{1}{|\lambda_1|}$$

da cui $\left(\frac{1}{|\lambda_1|}-\varepsilon\right)^n \underset{\text{per } \infty\, n}{\ll_\varepsilon} |a_n| \ll_\varepsilon \left(\frac{1}{|\lambda_1|}+\varepsilon\right)^n$ (se $\lambda_1 \neq 0$)

Se però il numeratore s'annullasse in λ_1 (e questo dipende dalle $\underline{c}$) allora il raggio sarà $|\lambda_2|$ ... e così via.

<u>Esempio</u>: $\begin{cases} a_{n+2} - 3a_{n+1} + 2a_n = (1/2)^n \\ a_0 = c_0, \quad a_1 = c_1 \end{cases}$

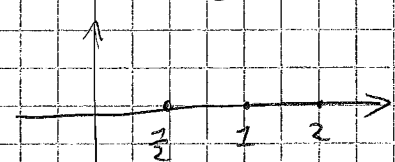
$F(z) = \sum_{n=0}^{\infty} a_n z^n$; $G(z) = \sum_{n=0}^{\infty} z^n/2^n = 1/(1-z/2) = 2/(2-z)$

$F(z) - c_0 - c_1 z - 3z(F(z) - c_0) + 2z^2 F(z) = 2z^2/(2-z)$

$$\Rightarrow F(z) = \frac{\frac{2z^2}{2-z} + c_0(1-3z) + c_1 z}{1-3z+2z^2}$$

$G(z) = 2/(2-z)$ ha raggio 2, $(1-3z+2z^2) = (1-z)(1-2z)$ radici $\frac{1}{2}$ e 1.

$\Rightarrow$ F ha $\rho = \frac{1}{2}$, a meno che il numeratore s'annulli in $1/2$



quindi se $-\frac{1}{2}c_0 + \frac{1}{2}c_1 + \frac{1}{3} \neq 0$ $\quad (2-\varepsilon)^n \underset{\infty\, n}{\ll_\varepsilon} |a_n| \ll_\varepsilon (2+\varepsilon)^n$

Se $-\frac{1}{2}c_0 + \frac{1}{2}c_1 + \frac{1}{3} = 0$

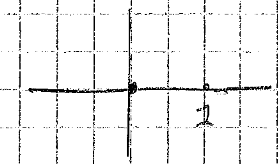
allora se poi il numeratore <u>NON</u> si annulla in 1, ovvero se

se $\begin{cases} -\frac{c_0}{2} + \frac{c_1}{2} + \frac{1}{3} = 0 \\ -2c_0 + c_1 + 2 \neq 0 \end{cases}$ $\quad (1-\varepsilon)^n \ll_\varepsilon |a_n| \ll_\varepsilon (1+\varepsilon)^n$

se però $\begin{cases} -\frac{c_0}{2} + \frac{c_1}{2} + \frac{1}{3} = 0 \\ 2c_0 + c_1 + 2 = 0 \end{cases}$ allora $\rho = 2$ e $\left(\frac{1}{2}-\varepsilon\right)^n \ll_\varepsilon |a_n| \ll_\varepsilon \left(\frac{1}{2}+\varepsilon\right)^n$

$(c_0 = 4/3, \; c_1 = 2/3) \Leftarrow$ in tal caso $F(z) = \dfrac{8/3}{2-z}$

-) Sviluppo di $\frac{e^z}{z}$ in 1 : raggio?



$z = 1+w \quad f(z) = \frac{e^z}{z} = \frac{e^{1+w}}{1+w} = e\cdot\frac{e^w}{1+w}$

$$= e\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^n w^n\right)$$

$$= \sum_{n=0}^{\infty}\left[\sum_{\substack{a,b\ge 0\\ a+b=n}} \frac{(-1)^b e}{a!}\right] w^n$$

raggio? distanza dalle singolarità, $\Rightarrow$ 1

oppure $\overline{\lim_{n\to\infty}}\left| e\sum_{a=0}^{n}\frac{(-1)^a}{(n-a)!}\right|^{\frac{1}{n}} = 1$

-) Sviluppo di $\frac{ch(z)}{z^2+i}$ in 0 : $\frac{1}{i+z^2}\cdot\sum_{n=0}^{\infty}\frac{\delta_2(n)\, z^n}{n!}$ $\qquad \delta_2(n) = \begin{cases} 1 & n \text{ pari} \\ 0 & n \text{ dispari}\end{cases}$

$$= \frac{1}{i}\,\frac{1}{1+iz^2}\sum_{n=0}^{\infty}\delta_2(n)\frac{z^n}{n!}$$

$$= -i\left(\sum_{n=0}^{\infty} i^n z^{2n}\right)\left(\sum_{n=0}^{\infty}\delta_2(n)\frac{z^n}{n!}\right)$$

$$= -i\left(\sum_{n=0}^{\infty} i^{n/2}\delta_2(n)\, z^n\right)\left(\sum_{n=0}^{\infty}\delta_2(n)\frac{z^n}{n!}\right)$$

$$= -i\sum_{n=0}^{\infty}\underbrace{\left(\sum_{a+b=n}\frac{i^{a/2}}{b!}\delta_2(a)\,\delta_2(b)\right)}_{=0 \text{ se } n \text{ dispari.}} z^n$$

$=$

$$= -i\sum_{n=0}^{\infty}\left(\sum_{a+b=n}\frac{i^a}{(2b)!}\right) z^{2n}$$ raggio 1.

-) $F(z) := \sum_{n=1}^{\infty}\frac{z^n}{1-z^n}$

ogni $\frac{z^n}{1-z^n}$ è <u>olo</u> in $D(0,1)$

Inoltre se $|z| \le r < 1$, allora $\left|\frac{z^n}{1-z^n}\right| \le \frac{r^n}{1-r^n} \le 2r^n$ se n suff. grande,

$\Rightarrow$ la Σ <u>conv.</u> unif. nei compatti di $D(0,1)$ $\Rightarrow$

$F(z)$ è <u>olomorfa</u> in <u>$D(0,1)$</u>.

oss che F ha ∞ singolarità in $\mathbb{T}$

Infatti, le ho in ogni z: $z^k=1$ per qualche k intero. ($\Rightarrow$ tutti gli interi ciclotomici).

Infatti:

fisso p primo (*) per $|z|<1$, ho

$$F(z) = \sum_{\ell=1}^{\infty} \frac{z^{\ell p}}{1-z^{\ell p}} + \sum_{\substack{n=1\\ p\nmid n}}^{\infty} \frac{z^n}{1-z^n}$$

ma se $p\nmid n$, allora se $z = e^{\frac{2\pi i}{p}\cdot j}$ $j=1,\dots,p-1$ coprimo con p

$$|1-z^n| = |z^{n/2}|\,|z^{n/2}-z^{-n/2}| = 2\left|\sin\left(\frac{\pi n j}{p}\right)\right| > c > 0 \quad \text{unif. in } n,$$

se $p\nmid n$, perché $nj \bmod p$ ha solo # finito di valori e nessuno è 0.

Quindi $\displaystyle\sum_{\substack{n=1\\ p\nmid n}}^{\infty} \frac{z^n}{1-z^n}$ converge in $e^{\frac{2\pi i j}{p}}$.

MA $\displaystyle\sum_{\ell=1}^{\infty} \frac{z^{\ell p}}{1-z^{\ell p}}$ evidentemente NON converge lì. $\Rightarrow$ $F(z)$ ha singolarità in ogni punto di quel tipo.

$$\Rightarrow F(z) = \sum_{n=1}^{\infty} z^n \sum_{m=0}^{\infty} z^{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} z^{mn} = \sum_{n=1}^{\infty} \underbrace{d(n)}\, z^n$$

dove $d(n) := \displaystyle\sum_{k|n} 1$ (funzione dei divisori).

Oggetto 1 $\Rightarrow$ $(1-\varepsilon)^n \underset{\text{per } \infty\, n}{\ll_{\varepsilon}} d(n) \ll_{\varepsilon} (1+\varepsilon)^n$

QUASI BANALE

BANALE

(*) la tesi è vera $\forall$ intero ciclotomico, ma è un po' più difficile da dimostrare

NON FATTO!

Qualche lemma sulle Σ di potenze. (21)

Lemma 1 Supp. $F(z) = \sum_{n=0}^{\infty} a_n z^n$, $G(z) = \sum_{n=1}^{\infty} b_n z^n$ con raggio 1, $\lim_{z\to 1^-} G(z) = \infty$.

Supp. $\underline{a_n, b_n > 0}$ e $a_n \sim b_n$ per $n \to \infty$. Allora $\underline{F(z) \underset{z\to 1^-}{\sim} G(z)}$.

Dim. Dato $\varepsilon > 0$, $\exists$ N t.c. $\frac{a_n}{b_n} < 1+\varepsilon$ per $n > N$.

Allora $F(z) - G(z) = \sum_{n\le N} (a_n - b_n) z^n + \sum_{n > N} (a_n - b_n) z^n$

$\Rightarrow$ $z \in (0,1)$

$$|F(z) - G(z)| \le \sum_{n\le N} |a_n - b_n| + \varepsilon \sum_{n>N} b_n z^n \le \sum_{n\le N} |a_n - b_n| + \varepsilon \sum_{n\le N} b_n + \varepsilon G(z)$$

$$\Rightarrow \quad \left|\frac{F(z)}{G(z)} - 1\right| \le \left(\sum_{n\le N} |a_n - b_n| + \varepsilon \sum_{n\le N} b_n\right) \frac{1}{G(z)} + \varepsilon$$

Ma $G(z) \to \infty$ per $z \to 1$, (perché b_n sono $\underline{\ge 0}$) quindi $\exists\, \delta$ t.c.

$$|z-1| < \delta \quad \Rightarrow \quad \left|\frac{F(z)}{G(z)} - 1\right| < 2\varepsilon.$$ ∎

In questa forma è di difficile applicazione, ma:

Lemma 2 Supp. $F(z) = \sum_{n=0}^{\infty} a_n z^n$, $G(z) = \sum_{n=1}^{\infty} b_n z^n$ con raggio 1 e $\lim_{z\to 1^-} G(z) = \infty$.

Supp. $\underline{a_n, b_n \ge 0}$, e $A(n) := \sum_{m\le n} a_m$, $B(n) := \sum_{m\le n} b_m$ con $A(n) \underset{n\to\infty}{\sim} B(n)$.

Allora $F(z) \underset{z\to 1^-}{\sim} G(z)$.

Dim Sia $\tilde{F}(z) := \sum_{n=0}^{\infty} A(n) z^n$, $\tilde{G}(z) = \sum_{n=0}^{\infty} B(n) z^n$

Dal lemma $\tilde{F}(z) \underset{z\to 1}{\sim} \tilde{G}(z)$.

Ma $\tilde{F}(z) = \frac{1}{1-z} F(z)$ e $\tilde{G}(z) = \frac{1}{1-z} G(z)$ $\Rightarrow$ Tesi ∎

Oss si usa la positività di A e B, non di a_n e b_n. $\Rightarrow$

se $a_n = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots\}$ va Bene.

Applicazione: $F(z) = \sum_{m=0}^{\infty} z^{2^m} = \sum_{n=0}^{\infty} \delta_2(n) z^n$ con $\delta_2(n) = \begin{cases} 1 & \text{se } n = 2^k \\ 0 & \text{altrimenti.} \end{cases}$

raggio è 1.

Allora $\sum_{m\le n} \delta_2(m) = \sum_{2^k \le n} 1 = [\lg_2 n]$

quindi $G(z) = \sum_{n=1}^{\infty} \frac{1}{n \lg 2} z^n$

allora $\sum_{m \le n} \frac{1}{m \lg 2} \sim \frac{1}{\lg 2} \lg n = \lg_2 n \Rightarrow$

$F(z) \underset{z \to 1}{\sim} G(z) = \frac{-1}{\lg 2} \lg(1-z)$!!

oss. $\sum_{m \le n} d(m) \sim n \lg n$ e $\sum_{m \le n} \lg m \sim n \lg n$

quindi

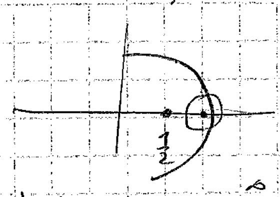
$$\sum_{k=1}^{\infty} \frac{z^k}{1-z^k} = \sum_{n=1}^{\infty} d(n) z^n \underset{\substack{z \to 1 \\ \text{Lemma } 2}}{\sim} \sum_{n=1}^{\infty} \lg n \, z^n \underset{\text{Lemma } 1}{\sim} \sum_{n=1}^{\infty} \left(\sum_{m \le n} \frac{1}{m} \right) z^n = \frac{1}{1-z} \sum_{n=1}^{\infty} \frac{1}{n} z^n = \frac{\lg(1-z)}{1-z}$$

(Pringsheim)

Lemma sia $F(z) = \sum_{n=0}^{\infty} a_n z^n$, raggio 1, $a_n \ge 0$. Allora F NON ha prolungamento analitico in $U(1)$ (intorno aperto di 1).

Dim supp. che $F(z)$ sia olomorfa in $U(1)$. quindi Σ di potenze

$$F(z) = \sum_{k=0}^{\infty} \frac{F^{(k)}(\frac{1}{2})}{k!} \left(z - \frac{1}{2}\right)^k$$ analitica in $\frac{1}{2}$. ha raggio $> \frac{1}{2}$.



ma $F^{(k)}(\frac{1}{2}) = \sum_{n=k}^{\infty} a_n \frac{n!}{(n-k)!} \left(\frac{1}{2}\right)^{n-k}$

$\Rightarrow F(z) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_n \frac{n!}{(n-k)!\,k!} \left(\frac{1}{2}\right)^{n-k} \left(z - \frac{1}{2}\right)^k$

prendo $z = 1+\varepsilon$ con $\varepsilon > 0$ e piccolo perché $z - \frac{1}{2} = \frac{1}{2} + \varepsilon$ sia entro il raggio di convergenza allora

$F(1+\varepsilon) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_n \binom{n}{k} \left(\frac{1}{2}\right)^{n-k} \left(\frac{1}{2} + \varepsilon\right)^k$ ma $a_n \ge 0 \Rightarrow$ le Σ commutano

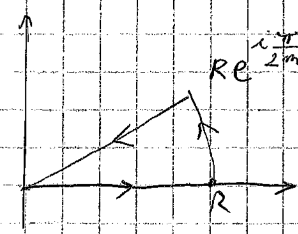
$$= \sum_{n=0}^{\infty} \sum_{0 \le k \le n} a_n \binom{n}{k} \left(\frac{1}{2}\right)^{n-k} \left(\frac{1}{2} + \varepsilon\right)^k \underset{\text{BINOMIALE}}{=} \sum_{n=0}^{\infty} a_n \left(\frac{1}{2} + \frac{1}{2} + \varepsilon\right)^n = \sum_{n=0}^{\infty} a_n (1+\varepsilon)^n$$ contro

il fatto che il raggio è 1 ∎

→ Calcolare $\int_0^{+\infty} \cos(x^m)\, dx$ e $\int_0^{\infty} \sin(x^m)\, dx$, dove $m \in \mathbb{R}$, $\underline{\underline{m > 1}}$.

<u>oss</u> che sono Re e Im di $\int_0^{\infty} e^{ix^m}\, dx$

Sia $R > 0$, "grande". quindi



allora $0 = \int_{\triangle} = \int_{\rightarrow} - \int_{\nearrow} + \int_{\curvearrowleft}$

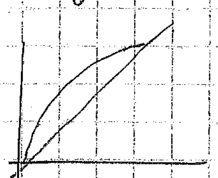
ma $\int_{\rightarrow} = \int_0^R e^{ix^m}\, dx$, quindi

e $\int_{\nearrow} = \int_0^R e^{i(\rho e^{\frac{\pi i}{2m}})^m} e^{\frac{\pi i}{2m}}\, d\rho = e^{\frac{\pi i}{2m}} \int_0^R e^{-\rho^m}\, d\rho$

e $\int_{\curvearrowleft} = \int_0^{\frac{\pi}{2m}} e^{i(Re^{i\theta})^m}\, iRe^{i\theta}\, d\theta$

$\Rightarrow \left| \int_{\curvearrowleft} \right| \leq R \int_0^{\frac{\pi}{2m}} e^{-R^m \sin(m\theta)}\, d\theta = \frac{R}{m} \int_0^{\frac{\pi}{2}} e^{-R^m \sin\theta}\, d\theta$

ma



$\sin\theta \geq \frac{2}{\pi}\theta$

$$\leq \frac{R}{m} \int_0^{\frac{\pi}{2}} e^{-\frac{2}{\pi}R^m \theta}\, d\theta = \frac{2R}{m\pi} \frac{1}{R^m} \left(1 - e^{-R^m}\right)$$

quindi per $R \to \infty$ (qui uso $m > 1$)

$$\int_0^R e^{ix^m}\, dx = e^{\frac{\pi i}{2m}} \int_0^R e^{-\rho^m}\, d\rho + O\left(\frac{1}{R^{m-1}}\right)$$

ovvero $\int_0^{\infty} e^{ix^m}\, dx = e^{\frac{\pi i}{2m}} \int_0^{\infty} e^{-\rho^m}\, d\rho$ $\quad (*)$

se $\underline{m=2}$, si sa che $\int_0^{+\infty} e^{-\rho^2}\, d\rho = \frac{\sqrt{\pi}}{2} \Rightarrow \int_0^{\infty} e^{ix^2}\, dx = (1+i)\frac{\sqrt{\pi}}{2\sqrt{2}}$.

per m generale, la $(*)$ dà

$$\int_0^\infty e^{ix^m}\,dx = e^{\frac{\pi i}{2m}}\int_0^\infty e^{-\rho^m}\,d\rho \underset{\substack{\rho^m = u \\ \rho = u^{\frac{1}{m}}}}{=} \frac{e^{\frac{\pi i}{2m}}}{m}\int_0^\infty e^{-u}\, u^{\frac{1}{m}}\,\frac{du}{u} = \frac{e^{\frac{\pi i}{2m}}}{m}\,\Gamma\left(\frac{1}{m}\right)$$

Verificare che $f(z) = \sum_{n\in\mathbb{Z}} e^{-n^2 z}$ è olomorfa in $\mathrm{Re}\, z > 0$.

Proof in $\mathrm{Re}\, z > \varepsilon > 0$ si ha

$$|e^{-n^2 z}| = e^{-n^2 \mathrm{Re}\, z} \leq e^{-\varepsilon n^2}$$

e $$\sum_{n\in\mathbb{Z}} e^{-\varepsilon n^2} = 1 + 2\sum_{n=1}^\infty e^{-\varepsilon n^2} < +\infty$$

$\Rightarrow$ per Test di Weierstrass la Σ conv unif in $\mathrm{Re}\, z > \varepsilon > 0$, $\forall \varepsilon$.

$\Rightarrow$ f è olomorfa.

[Argomento standard: C.U $\Rightarrow$ f continua. $\forall \gamma$ contorno chiuso in $\mathrm{Re}\, z > 0$, si ha che $d(\gamma, \text{asse } I_{x=0}) > 0$ $\Rightarrow$ la conv è unif in γ $\Rightarrow$ $\int_\gamma f(z)\,dz = \int_\gamma \Sigma\, dz = \Sigma \int_\gamma e^{-n^2 z}\,dz \underset{e^{-n^2 z} \text{ è olo}}{=} \Sigma\, 0 = 0$

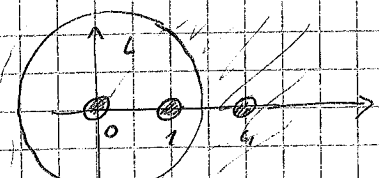
$\Rightarrow$ Morera $\square$

Verificare che $f(z) = \sum_{n\in\mathbb{Z}} \frac{1}{z - n^2}$ è olomorfa in $\mathbb{C} \setminus \{0^2, 1^2, 2^2, 3^2, \dots\}$

Proof sia $D = \mathbb{C} \setminus \{0, 1, 2^2, 3^2, \dots\}$. D è aperto.

Fisso $L > 0$ (grande) sia $D_L = D(0, L) \setminus \{0^2, 1^2, 2^2, 3^2, \dots\}$

$D_4 =$



In D_L si ha $|z - n^2| \geq |n^2 - |z|| \underset{\text{se } n^2 > L}{\geq} n^2 - |z| > n^2 - L$

quindi $$\sum_{n\in\mathbb{Z}} \frac{1}{z-n^2} = \sum_{|n| \leq .} \frac{1}{z - n^2} + \sum_{|n|^2 > L} \frac{1}{z - n^2}$$

e $$\left|\sum_{|n|^2 > L} \frac{1}{z - n^2}\right| \leq \sum_{n^2 > L} \frac{1}{n^2 - L} < \infty$$ unif in D_L

Quindi $\underline{\text{tutte}}$ le Σ conv. unif. in D_L.

$\Rightarrow$ (continuità + $\int_{\gamma \in \text{chiuso}}$ + Morera) è olomorfa in D_L

$\Rightarrow$ lo è in $\bigcup_L D_L = D$.

$\longmapsto$

Calcolare $\displaystyle\int_\gamma f(z)\,dz$ dove γ è (circonferenza di centro 0 e raggio R) con $R \notin \{n^2\}$.

$$= 2\pi i \cdot \sum_{\substack{n\in\mathbb{Z}\\ n^2<R}} 1 = 2\pi i\left(1+2\lfloor\sqrt{R}\rfloor\right)$$

$\longmapsto$

$$\sum_{n=1}^{\infty}\left[\sin\left(\frac{z}{n}\right)-\frac{z}{n}\right] \qquad \text{Se } D_L = D(0,L)\,,\ L>0$$

$$|\sin w - w| = \left|\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}w^{2k+1} - w\right| = \left|\sum_{k=1}^{\infty}(-1)^k\frac{w^{2k+1}}{2k+1}\right| \leq \sum_{k=1}^{\infty}\frac{|w|^{2k+1}}{(2k+1)!}$$

$$= \operatorname{Sh}|w| - |w| = \left(\frac{1}{3!}+o(1)\right)|w|^3$$

$\Rightarrow \exists\ \varepsilon>0$ t.c. $|\sin w - w| \leq |w|^3$ se $|w|<\varepsilon$.

Quindi se $n \geq L/\varepsilon$, si ha $\left|\frac{z}{n}\right| \leq \frac{L}{n} \leq \varepsilon \Rightarrow$

$$\sum_{n=1}^{\infty}[\ ] = \sum_{n=1}^{\lfloor L/\varepsilon\rfloor}[\ ] + \underbrace{\sum_{n=\lfloor L/\varepsilon\rfloor+1}^{\infty}[\ ]}$$

$$|\quad| \leq \sum_{n=\lfloor L/\varepsilon\rfloor+1}^{\infty}|\ | \leq \sum_{n=\lfloor L/\varepsilon\rfloor+1}^{\infty}\frac{L^3}{n^3} \leftarrow \text{converge}$$

$\Rightarrow$ $\underline{\text{c. unif.}}$ in D_L $\Rightarrow$ olomorfa in D_L $\forall L$ $\Rightarrow$ olomorfa in $\mathbb{C}$.

$\longmapsto$

$$\prod_{n=1}^{\infty}\cos\left(\frac{z}{n}\right) \quad \text{converge? olomorfa?}$$

Γ la convergenza è assoluta $\Rightarrow$ zeri vengono da zeri dei fattori,

$\Rightarrow \{\text{zeri}\} = \displaystyle\bigcup_{n=1}^{\infty}{}^{\!\!\cdot}\ \{n\cdot\pi(\tfrac{1}{2}+k),\ k\in\mathbb{Z}\}$ $\quad \dot\cup$ = unione con molteplicità.

$=$ ogni zero è $z=\frac{\pi}{2}\cdot h$ con h intero, che ha molteplicità $= d(|h'|)$

dove h' = parte dispari di h e $d(\cdot)$ = funzione dei $\underline{\text{divisori}}$.

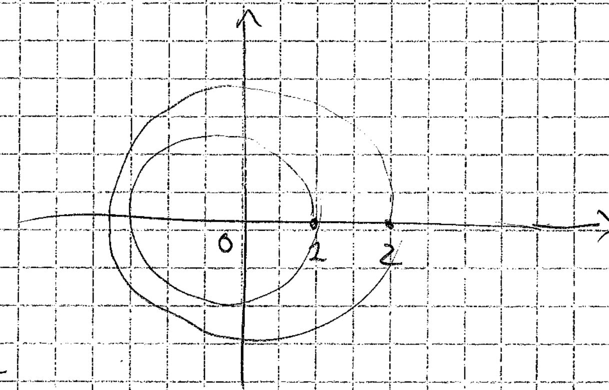
$\vdash$ A un compatto in $\mathbb{C}$, $\{a_n\}_{n=1}^{\infty}$ denso in A.

$$\sum_{n=1}^{\infty} \frac{2^{-n}}{z-a_n}$$ converge in $\mathbb{C}\setminus A$, e la convergenza è uniforme nei compatti di $\mathbb{C}\setminus A$ $\Rightarrow$ è olomorfa in $\mathbb{C}\setminus A$.

ogni a_n è una singolarità per la Σ (difficile da provare se posto $E := \{a_n\}_1^{\infty}$, si ha che E' ($= \{$punti di acc. di $E\}$) non è di punti isolati, ovvero se $E'' \neq \emptyset$.

H. Laurent

$$f(z) = \frac{1}{z(z-1)(z-2)}$$



•) Sviluppo Laurent in $A_{1,2}(0)$

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \quad \text{convergente in } 1<|z|<2$$

$$a_n = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z^{n+1}}\, dz \qquad n\in\mathbb{Z}, \qquad 1<\rho<2$$

$$\frac{1}{(z-1)(z-2)} = \frac{+1}{(2-z)(1-z)} = \frac{-1}{2}\,\frac{1}{1-\frac{z}{2}}\cdot\frac{1}{z}\,\frac{1}{1-\frac{1}{z}}$$

$$= -\frac{1}{2z}\,\frac{1}{1-\frac{z}{2}}\cdot\frac{1}{1-\frac{1}{z}} = -\frac{1}{2z}\cdot\underbrace{\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n}_{|z|<2}\;\underbrace{\sum_{m=0}^{\infty}\left(\frac{1}{z}\right)^m}_{|z|>1}$$

$$= -\frac{1}{2z}\sum_{k\in\mathbb{Z}}\left(\sum_{\substack{n,m\geq 0\\ n-m=k}}\frac{1}{2^n}\right)z^k$$

$$= -\frac{1}{2z}\sum_{k\in\mathbb{Z}}\left(\sum_{\substack{m\geq 0\\ k+m\geq 0}}\frac{1}{2^{k+m}}\right)z^k$$

$$= -\frac{1}{2z}\sum_{k\in\mathbb{Z}}\left(\sum_{m\geq \max(0,-k)}\frac{1}{2^{k+m}}\right)z^k$$

$$= -\frac{1}{2z}\left[\sum_{k=0}^{+\infty}\left[\sum_{m\geq 0}\frac{1}{2^{k+m}}\right]z^k + \sum_{k=1}^{\infty}\left[\sum_{m=k}^{\infty}\frac{1}{2^{m-k}}\right]z^{-k}\right]$$

$$= -\frac{1}{2z}\left[\sum_{k=0}^{\infty}\frac{2}{2^k}z^k + \sum_{k=1}^{\infty}2\,z^{-k}\right] = -\frac{1}{z}\left[\sum_{k=0}^{\infty}\frac{1}{2^k}z^k + \sum_{k=1}^{\infty}z^{-k}\right]$$

$$\Rightarrow f(z) = -\frac{1}{z^2}\left[\quad\right] = -\frac{1}{z^2}\left[\dots z^{-3} + z^{-2} + z^{-1} + 1 + \frac{z^1}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \frac{z^4}{2^4} + \dots\right]$$

$$= \dots - z^{-5} - z^{-4} - z^{-3} + z^{-2} + \frac{z^{-1}}{2} + \frac{1}{4} + \frac{z}{8} + \frac{z^2}{16} + \frac{z^3}{32} + \dots$$

quindi $= \sum_{n\in\mathbb{Z}} a_n z^n$ con

$$a_n = \begin{cases} -1 & n\leq -3 \\ 1 & n=-2 \\ 1/2 & n=-1 \\ \frac{1}{2^{n+2}} & n\geq 0 \end{cases}$$

•) Sviluppo in $A_{0,1}(0)$

$$f(z) = \frac{1}{z}\,\frac{1}{z-1}\,\frac{1}{z-2} = \frac{1}{2z}\,\frac{1}{1-z}\,\frac{1}{1-\frac{z}{2}} = \frac{1}{2z}\underbrace{\sum_{n=0}^{\infty} z^n}_{\text{conv. } |z|<1}\ \underbrace{\sum_{m=0}^{\infty} \frac{z^m}{2^m}}_{\text{conv. } |z|<2}$$

$$= \frac{1}{2z}\sum_{k=0}^{\infty}\left(\sum_{\substack{n,m\geq 0\\ n+m=k}}\frac{1}{2^m}\right)z^k = \frac{1}{2z}\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\frac{1}{2^m}\right)z^k$$

$$= \frac{1}{2z}\sum_{k=0}^{\infty}\frac{1-\frac{1}{2^{k+1}}}{1-\frac{1}{2}}\,z^k = \frac{1}{z}\sum_{k=0}^{\infty}\left(1-\frac{1}{2^{k+1}}\right)z^k = \frac{1}{2z} + \sum_{k=0}^{\infty}\left(1-\frac{1}{2^{k+2}}\right)z^k$$

•) Sviluppo in $A_{2,\infty}(0)$

$$f(z) = \frac{1}{z(z-1)(z-2)} = \frac{1}{z^3}\,\frac{1}{1-\frac{1}{z}}\,\frac{1}{1-\frac{2}{z}} = \frac{1}{z^3}\underbrace{\underbrace{\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n}_{\text{conv. } |z|>1}\ \underbrace{\sum_{m=0}^{\infty}\frac{2^m}{z^m}}_{\text{conv. } |z|>2}}_{\text{conv. } |z|>2}$$

$$= \frac{1}{z^3}\sum_{k=0}^{\infty}\left(\sum_{\substack{n,m\geq 0\\ n+m=k}} 2^m\right)z^{-k} = \frac{1}{z^3}\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} 2^m\right)z^{-k}$$

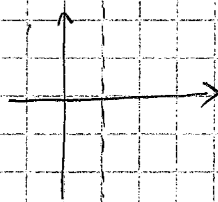
$$= \frac{1}{z^3}\sum_{k=0}^{\infty}\left(2^{k+1}-1\right)z^{-k} = \sum_{k=3}^{\infty}\left(2^{k-2}-1\right)z^{-k}$$

Metodo più veloce: in $A_{1,2}(0)$: $\frac{1}{z-1}\cdot\frac{1}{z-2} = \frac{1}{z-2}-\frac{1}{z-1}$

$$= -\frac{1}{2}\,\frac{1}{1-\frac{z}{2}} - \frac{1}{z}\,\frac{1}{1-\frac{1}{z}} = -\frac{1}{2}\underbrace{\sum_{k=0}^{\infty}\frac{z^k}{2^k}}_{|z|<2} - \frac{1}{z}\underbrace{\sum_{k=0}^{\infty}\frac{1}{z^k}}_{|z|>1} \Rightarrow \text{Tesi.}$$

-) $y>0$, $c>0$ $\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s^2}\, ds = \log^+ y := \max(0,\log y) = \begin{cases}\log y, & y\ge 1\\ 0 & 0<y<1\end{cases}$

Proof

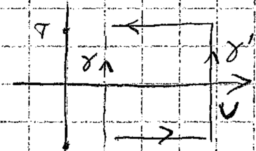


$\left|\frac{y^s}{s^2}\right| \underset{\text{su } c+i\mathbb{R}}{=} \frac{y^c}{c^2+t^2}$ ⇒ ass. integrabile.

$$\Rightarrow \int_{c-i\infty}^{c+i\infty}\frac{y^s}{s^2}\,ds = \lim_{T\to+\infty}\int_{c-iT}^{c+iT}\frac{y^s}{s^2}\,ds$$

supp. $y\in(0,1)$ allora $y^\sigma\to 0$ se $\sigma\to+\infty$.

⇒



$\gamma\cup-\gamma'$ è cammino chiuso orientato in regione sempl. connessa dove $\frac{y^s}{s^2}$ è olomorfa ⇒

$$\int_{\gamma\cup-\gamma'}=0 \Rightarrow \int_{c-iT}^{c+iT} = \int_{\gamma'} = \int_{\rightarrow} + \int_{U-iT}^{U+iT} + \int_{\leftarrow}$$

ma $\left|\int_{\rightarrow}\right| = \left|\int_c^U \frac{y^{\sigma-iT}}{(\sigma-iT)^2}\,d\sigma\right| \le \int_c^U\frac{y^\sigma}{\sigma^2+T^2}\,d\sigma \le \frac{1}{T^2}\int_c^U y^\sigma\, d\sigma$

$$\underset{y\le 1}{\le} \frac{1}{T^2}\cdot\int_c^U d\sigma \underset{c>0}{\le} \frac{U}{T^2}$$

lo stesso vale per $\int_{\rightarrow}$.

e $\left|\int_{U-iT}^{U+iT}\right| = \left|\int_{-T}^{T}\frac{y^{U+it}}{(U+it)^2}\, i\,dt\right| \le \int_{-T}^{T}\frac{y^U}{U^2+t^2}\,dt \le y^U\int_{-T}^{T}\frac{dt}{U^2+t^2} \le \frac{y^U}{U}\pi$

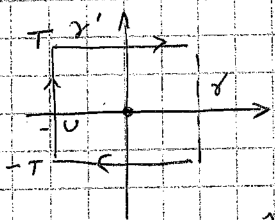
quindi:

$\left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\right| \le \frac{U}{\pi T^2}+\frac{y^U}{2U}$ $\quad\forall\, U, T.$ prendo $U=T$

$$\le \frac{1}{\pi T}+\frac{y^T}{2T} \xrightarrow[T\to\infty]{} 0$$ (essendo $y\in(0,1)$).

Supp. $y > 1$. Allora $y^{\sigma} \to 0$ se $\sigma \to \underline{\underline{-\infty}}$.

Quindi



$$\frac{1}{2\pi i}\int_{\gamma \cup -\gamma'} = \operatorname*{Res}_{s=0} \frac{y^s}{s^2} = \operatorname*{Res}_{s=0}\left[\frac{e^{s\log y}}{s^2}\right] = \operatorname*{Res}_{s=0}\left[\frac{1+s\log y + O(s^2)}{s^2}\right] = \log y$$

quindi $\displaystyle \frac{1}{2\pi i}\int_{c-iT}^{c+iT} = \log y + \frac{1}{2\pi i}\int_{-\gamma'}$

ma $\displaystyle \int_{-\gamma'} = \int_{\leftarrow} + \int_{-U-iT}^{-U+iT} + \int_{\rightarrow}$

e $\displaystyle \left|\int_{\leftarrow}\right| = \left|\int_{c}^{-U} \frac{y^{\sigma-iT}}{(\sigma-iT)^2}\,d\sigma\right| \le \int_{-U}^{c}\frac{y^{\sigma}}{\sigma^2+T^2}\,d\sigma \le \frac{y^c}{T^2}(c+U)$

$\displaystyle \int_{\rightarrow}$ come sopra

e $\displaystyle \left|\int_{-U-iT}^{-U+iT}\right| = \left|\int_{-T}^{T}\frac{y^{-U+it}}{(-U+it)^2}\, i\,dt\right| \le y^{-U}\int_{-T}^{T}\frac{dt}{U^2+t^2} \le \frac{y^{-U}}{U}\pi$

quindi $\displaystyle \left|\frac{1}{2\pi i}\int_{-\gamma'}\right| \le \frac{y^c}{\pi T^2}(c+U) + \frac{y^{-U}}{2U}$ $\quad\forall\, U$ e T.

Posto $U = T$

$$\le \frac{y^c}{\pi}\left(\frac{c}{T^2}+\frac{1}{T}\right) + \frac{y^{-T}}{2T} \longrightarrow \text{ se } T\to\infty.$$

•) Supp. $y=1$. Allora

$$\int_{c-iT}^{c+iT}\frac{y^s}{s^2}\,ds = \int_{c-iT}^{c+iT}\frac{ds}{s^2} = \frac{-1}{c+iT} + \frac{1}{c-iT} \xrightarrow[T\to\infty]{} 0.$$

∎

Variazione:

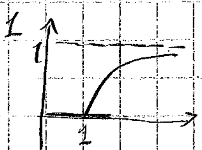
$$y>0,\quad c>0,\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s^m}\,ds=\frac{1}{(m-1)!}(\log y)^{m-1}$$

$m\geq 1$

(Intero)

Variazione:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s(s+1)}\,ds=\begin{cases}\left[\mathrm{Res}_{s=0}+\mathrm{Res}_{s=-1}\right]\left[\frac{y^s}{s(s+1)}\right]=1-\frac{1}{y} & y\geq 1\\ 0 & 0<y<1\end{cases}$$



è C^0

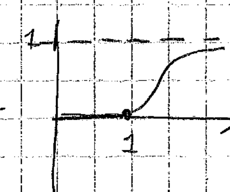
$$\mathrm{Res}_{s=0}=\lim_{s\to 0} s\cdot\frac{y^s}{s(s+1)}=1$$

Polo semplice

$$\mathrm{Res}_{s=-1}=\lim_{s\to -1}(s+1)\frac{y^s}{s(s+1)}=-\frac{1}{y}$$

Variazione:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s(s+1)(s+2)}\,ds=\begin{cases}\frac{1}{2}\left(1-\frac{1}{y}\right)^2 & y\geq 1\\ 0 & 0<y\leq 1\end{cases}$$



è C^1

Variazione:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s(s+1)\cdots(s+m)}\,ds=\begin{cases}\frac{1}{m!}\left(1-\frac{1}{y}\right)^m & y\geq 1\\ 0 & y\in(0,1)\end{cases}$$

è C^{m-1}

$m\geq 1$,

Intero

$$\int_{-\pi}^{\pi}\frac{8\cos(2\theta)-15\cos\theta+2}{48\cos 3\theta+104\cos 2\theta+252\cos\theta-221}\,d\theta$$

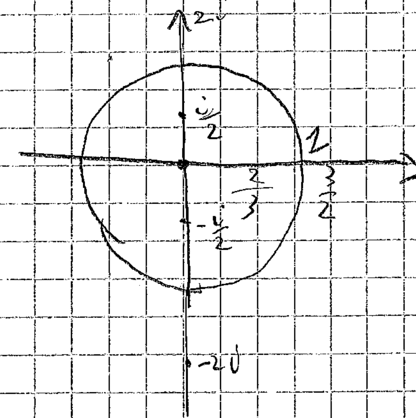
Pongo $z=e^{i\theta}$: z varia su ∂D

allora $2\cos\theta=z+\frac{1}{z}$, $2\cos 2\theta=z^2+\frac{1}{z^2}$, $2\cos 3\theta=z^3+\frac{1}{z^3}$

$$=\oint\frac{4\left(z^2+\frac{1}{z^2}\right)-\frac{15}{2}\left(z+\frac{1}{z}\right)+2}{24\left(z^3+\frac{1}{z^3}\right)-52\left(z^2+\frac{1}{z^2}\right)+126\left(z+\frac{1}{z}\right)-221}\,\frac{dz}{iz}$$

$$=\oint\frac{8z^4-15z^3+4z^2-15z+8}{48z^6-104z^5+252z^4-442z^3+252z^2-104z+48}\,\frac{dz}{i}$$

$$= \frac{1}{i}\int_{\bigcirc} \frac{\text{———}}{(4z^2+1)(z^2+4)(3z-2)(2z-3)}\, dz$$



$$= \frac{2\pi i}{i}\cdot\left[\operatorname{Res}(i/2) + \operatorname{Res}(-i/2) + \operatorname{Res}(2/3)\right]$$

$$= 2\pi\left[\frac{1+3i}{80} + \frac{1-3i}{80} + \frac{1}{40}\right]$$

$$= \frac{\pi}{10}.$$

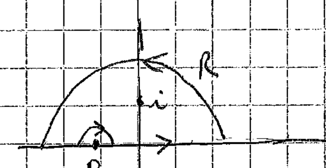
(+)

e.) $a \in \mathbb{R}$. Calcolare $\displaystyle\int_{\mathbb{R}} \frac{\cos x}{(x-a)(x^2+1)}\,dx$ (se $a \in S := \{\frac{\pi}{2}+k\pi, k \in \mathbb{Z}\}$ allora $\frac{\cos x}{x-a}$ è olo in a. $\Rightarrow$ ok. Se però $a \notin S$ allora $x=a$ è Polo, e $\displaystyle\int_{\mathbb{R}} := \lim_{\varepsilon \to 0} \int_{\mathbb{R}\setminus(a-\varepsilon,\, a+\varepsilon)}$ (Valor principale).

Oss

$$\int_{\mathbb{R}} \text{--} = \operatorname{Re} \int_{\mathbb{R}} \frac{e^{ix}}{(x-a)(x^2+1)}\,dx$$

quindi calcolo $P.P.\displaystyle\int_{\mathbb{R}} \frac{e^{ix}}{(x-a)(x^2+1)}\,dx$.



$\boxed{R > a}$

$$\int_{-\frown^{\varepsilon}_{a}\rightarrow} + \int_{\frown} = 2\pi i \operatorname*{Res}_{z=i} f(z) = 2\pi i \lim_{z \to i} (z-i)\frac{e^{iz}}{(z-a)(z^2+1)} = 2\pi i\,\frac{e^{-1}}{(i-a)2i} = \frac{\pi/e}{i-a}$$

e

$$\int_{\frown} = \int_0^{\pi} \frac{e^{iRe^{i\theta}}}{(Re^{i\theta}-a)(R^2e^{2i\theta}+1)}\, R i e^{i\theta}\, d\theta$$

$$\Rightarrow \left|\ \right| \le R\int_0^{\pi} \frac{e^{-R\sin\theta}}{(R-a)(R^2-1)}\, d\theta \le \frac{2R}{(R-a)(R^2-1)} \int_0^{\pi/2} 1 \, d\theta \ll \frac{1}{R^3} \to 0$$

Inoltre, $\displaystyle\int_{\curvearrowleft} \frac{e^{iz}}{(z-a)(z^2+1)}\,dz = ?$ $\qquad f(z) = \dfrac{e^{iz}}{z^2+1}$

allora $\displaystyle\int_{\curvearrowleft} \frac{f(z)}{z-a}\,dz = \int_{\curvearrowleft} \frac{f(a)}{z-a}\,dz + \int_{\curvearrowleft} \frac{f(z)-f(a)}{z-a}\,dz = I + II$

ma $\displaystyle\left|\frac{f(z)-f(a)}{z-a}\right| \xrightarrow[z\to a]{} |f'(a)|$ quindi $|II| \ll \varepsilon$ per $\varepsilon \to 0$.

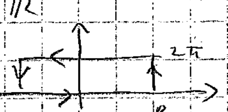
Invece, $\displaystyle\int_{\curvearrowleft} \frac{f(a)}{z-a}\,dz = -f(a)\int_0^{\pi} d\vartheta = \underline{\underline{-i\pi f(a)}}$

quindi, mandando $R\to\infty$ e $\varepsilon\to 0$, si ha

$$\text{P.P.}\int_{\mathbb{R}} \frac{e^{ix}}{(x-a)(x^2+1)}\,dx = \frac{\pi/e}{i-a} - \lim_{\varepsilon\to 0}\int_{\curvearrowleft} = \frac{\pi/e}{i-a} + i\pi f(a) = \frac{\pi/e}{i-a} + i\pi\frac{e^{ia}}{a^2+1}$$

$$\Rightarrow \text{P.P.}\int_{\mathbb{R}} \frac{\cos x}{(x-a)(x^2+1)}\,dx = \mathrm{Re}\Big[\quad\Big] = \frac{-a\,\pi/e}{a^2+1} - \frac{\pi\sin a}{a^2+1} = \frac{\pi}{a^2+1}\left(-\sin a - \frac{a}{e}\right)$$

$\displaystyle\int_{\mathbb{R}} R(e^x)\,dx$ o $\displaystyle\int_{\mathbb{R}} R(e^x)e^{ix}\,dx$ o $\displaystyle\int_{\mathbb{R}} R(e^x)\,x\,dx$:

con R razionale. Integrare su 

Calcolare $\displaystyle\int_0^{+\infty} \frac{\cos x}{\mathrm{ch}\,x}\,dx = \frac{1}{2}\,\mathrm{Re}\int_{\mathbb{R}} \frac{e^{ix}}{\mathrm{ch}\,x}\,dx$

$$\int \frac{e^{iz}}{\mathrm{ch}\,z}\,dz = 2\pi i\left[\mathrm{Res}_{z=i\frac{\pi}{2}}\left[\frac{e^{iz}}{\mathrm{ch}\,z}\right] + \mathrm{Res}_{z=\frac{3}{2}\pi i}\Big[\quad\Big]\right] = 2\pi i\left[\frac{e^{-\pi/2}}{\mathrm{sh}(i\frac{\pi}{2})} + \frac{e^{-\frac{3}{2}\pi}}{\mathrm{sh}(\frac{3}{2}\pi i)}\right]$$

$$= 2\pi\left[e^{-\pi/2} - e^{-\frac{3}{2}\pi}\right].$$

MA $\displaystyle\left|\int_{\uparrow}\right| = \left|\int_0^{2\pi} \frac{e^{i(R+iv)}}{\mathrm{ch}(R+iv)}\,i\,dv\right| \le \int_0^{2\pi} \frac{e^{-v}}{|\mathrm{ch}(R+iv)|}\,dv \le \int_0^{2\pi} \frac{2e^{-v}}{|e^R - e^{-R}|}\,dv \ll e^{-R}$ $\quad R\to+\infty$.

$$|\mathrm{ch}(R+iv)| = \frac{1}{2}\left|e^{R+iv} + e^{-R-iv}\right| \ge \frac{1}{2}\left|e^R - e^{-R}\right|$$

lo stesso $\int_{\downarrow}$.

Infine, $\int_{\leftarrow} = -\int_{-R}^{R} \frac{e^{i(x+2\pi i)}}{Ch(x+2\pi i)}\,dx = -e^{-2\pi}\int_{-R}^{R}\frac{e^{ix}}{ch\, x}\,dx$

$$\Rightarrow (1-e^{-2\pi})\int_{-R}^{R} \frac{e^{ix}}{ch\,x}\,dx = 2\pi\left[e^{-\frac{\pi}{2}} - e^{-\frac{3}{2}\pi}\right] + O(e^{-R})$$

$$\Rightarrow \int_{\mathbb{R}} \frac{e^{ix}}{ch\,x}\,dx = 2\pi\, e^{-\pi/2}\,\frac{[e^{-\pi}-1]}{[e^{-2\pi}-1]} = \frac{\pi}{ch\,\frac{\pi}{2}}$$

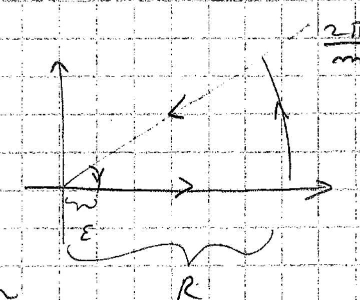
$$\Rightarrow \int_0^{+\infty}\frac{\cos x}{ch\,x}\,dx = \frac{\pi/2}{ch(\pi/2)}$$ ■

Provare: $\int_0^{+\infty}\frac{\cos(xy)}{ch\,x}\,dx = \frac{\pi/2}{ch(y\frac{\pi}{2})}$ $\quad\forall y\in\mathbb{R}$.

$\int_0^{+\infty} x^a R(x^m)\,dx$ $\quad\rightarrow$ Integrare su

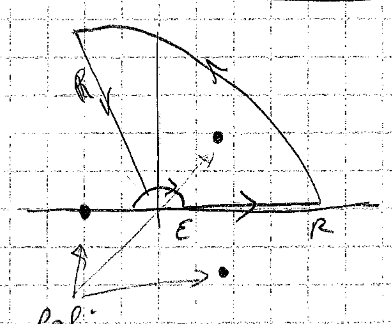
$a\in\mathbb{R}$, $\underline{m\geq 2}$ $\quad$ con $x^a = e^{a\lg x}$

(Attenzione al caso $m=2$) $\quad$ dove lg in vola principale.



$\int_0^{+\infty}\frac{x^a}{1+x^3}\,dx$ esiste per $\underline{-1<a<2}$

cammino γ:



$$\int_\gamma \frac{e^{a\lg z}}{1+z^3}\,dz = 2\pi i \operatorname*{Res}_{z=e^{i\pi/3}}\frac{e^{a\lg z}}{1+z^3} = 2\pi i\,\frac{e^{a\frac{i\pi}{3}}}{3e^{2i\pi/3}} = -2\pi i\, e^{(a+1)\frac{i\pi}{3}}$$

MA: $\int_{\rightarrow} = \int_\varepsilon^R \frac{x^a}{1+x^3}\,dx \rightarrow \int_0^\infty\frac{x^a}{1+x^3}\,dx$

$$\int_{\curvearrowleft} = \int_0^{2\pi/3}\frac{e^{a[\lg R + i\theta]}}{1+R^3e^{3i\theta}}\,R\,i\,e^{i\theta}\,d\theta$$

$$\Rightarrow \left| \int_{\gamma} \right| \le \frac{R^{a+1}}{R^3-1} \cdot \frac{2\pi}{3} \ll R^{a-2} \to 0.$$

$$\int_{\gamma} = -\int_0^{2\pi/3} \frac{e^{a(\log \varepsilon + i\vartheta)}}{1+\varepsilon^3 e^{3i\vartheta}} \varepsilon i e^{i\vartheta} \, d\vartheta$$

quindi $\left| \int_{\gamma} \right| \le \frac{\varepsilon^{a+1}}{1-\varepsilon^3} \cdot \frac{2\pi}{3} \ll \varepsilon^{a+1} \to 0.$

$$\int_{\searrow} = -\int_{\varepsilon}^{R} \frac{e^{a(\log x + i\frac{2\pi}{3})}}{1+x^3 e^{\frac{2\pi i}{3}3}} e^{i\frac{2\pi}{3}} dx = -e^{(a+1)i\frac{2\pi}{3}} \int_{\varepsilon}^{R} \frac{x^a}{1+x^3} dx$$

$$\Rightarrow \left(1-e^{(a+1)\frac{2\pi i}{3}}\right) \int_0^{\infty} \frac{x^a}{1+x^3} dx = -2\pi i \, e^{(a+1)\frac{\pi i}{3}}$$

$$\Rightarrow \int_0^{\infty} \frac{x^a}{1+x^3} dx = \frac{-2\pi i \, e^{(a+1)\pi i/3}}{1-e^{2(a+1)\pi i/3}} = \frac{-2\pi i}{e^{-(a+1)\frac{\pi i}{3}} - e^{(a+1)\frac{\pi i}{3}}} = \frac{\pi}{\sin\left((a+1)\frac{\pi}{3}\right)}$$

$$\int_0^{+\infty} x^a R(x)\, dx \quad \text{o} \quad \int_0^{+\infty} R(x)\, dx \quad \text{o} \quad \int_0^{+\infty} R(x) \log x \, dx$$

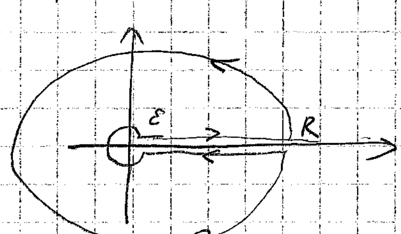
$a \notin \mathbb{Z}$

Sia $\mathrm{Log}_{(2\pi)} :=$ log olomorfo su $\mathbb{C} \setminus [0, +\infty)$

$$\mathrm{Log}_{(2\pi)}(z) = \log|z| + i\vartheta \quad \text{dove} \quad z = |z| e^{i\vartheta} \quad \vartheta \in (0, 2\pi)$$

Integrare $f(z) = e^{a \mathrm{Log}_{(2\pi)} z} R(z)$, $\mathrm{Log}_{(2\pi)}(z) R(z)$, $\left(\mathrm{Log}_{(2\pi)}(z)\right)^2 R(z)$

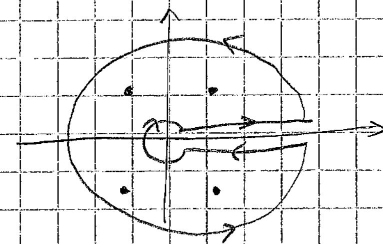
su



"Henkel"

esempio

$$\int_0^{+\infty} \frac{\lg x}{x^4+1}\, dx = ?$$



$$\int_\gamma (Lg_{2\pi} z)^2 \frac{dz}{1+z^4} = 2\pi i \sum_{k=0}^{3} \operatorname{Res}_{z=e^{\frac{\pi}{4}i + k i \frac{\pi}{2}}} \left(\right)$$

$$\int_{\rightarrow} = \int_\varepsilon^R \frac{(\lg x)^2}{1+x^4}\, dx$$

$$\int_{\leftarrow} = -\int_\varepsilon^R \frac{(\log x + 2\pi i)^2}{1+x^4}\, dx$$

$$\left|\int_{\circlearrowleft}\right| \le \left| \int_0^{2\pi} \frac{(\lg R + i\vartheta)^2}{1+R^4 e^{4i\vartheta}} R i e^{i\vartheta}\, d\vartheta \right| \le \frac{R(\lg^2 R + 4\pi^2)}{R^4-1} \cdot 2\pi \to 0$$

$$\left|\int_{\circlearrowright}\right| \le \left| -\int_0^{2\pi} \frac{(\lg \varepsilon + i\vartheta)^2}{1+\varepsilon^4 e^{4i\vartheta}} \varepsilon i e^{i\vartheta}\, d\vartheta \right| \le \varepsilon \frac{(\lg^2 \varepsilon + 4\pi^2)}{1-\varepsilon^4} \cdot 2\pi \to 0$$

Quindi passando al limite:

$$\int_0^{+\infty} \frac{(\lg x)^2 - (\lg x + 2\pi i)^2}{1+x^4}\, dx = 2\pi i \left[\left(\frac{\pi}{4} i\right)^2 \cdot \frac{1}{4 e^{3\frac{\pi}{4}i}} + \left(\frac{3}{4}\pi i\right)^2 \frac{1}{4 e^{3\cdot\frac{3\pi i}{4}}} + \left(\frac{5\pi i}{4}\right)^2 \cdot \frac{1}{4 e^{3\cdot\frac{5}{4}i\pi}} + \left(\frac{7}{4}\pi i\right)^2 \cdot \frac{1}{4 e^{3\cdot\frac{7}{4}\pi i}} \right]$$

$$= 2\pi i \cdot \frac{1}{4} \frac{\pi^2}{4^2} \cdot \frac{16+64i}{\sqrt{2}}$$

$$\Rightarrow -2\cdot 2\pi i \int_0^{+\infty} \frac{\lg x}{1+x^4}\, dx - (2\pi i)^2 \int_0^{+\infty} \frac{dx}{1+x^4} = 2\pi i \frac{\pi^2}{4^3} \cdot \frac{16+64i}{\sqrt{2}}$$

$$\Rightarrow \int_0^{+\infty} \frac{\lg x}{1+x^4}\, dx = -\pi i \int_0^{\infty} \frac{dx}{1+x^4} - \frac{\pi^2}{4} \cdot \frac{1-4i}{2\sqrt{2}}$$

$$\Rightarrow \int_0^{+\infty} \frac{\lg x}{1+x^4}\, dx = \operatorname{Re}\left(\int_0^{+\infty} \frac{\lg x}{1+x^4} \right) = \operatorname{Re}\left(\right) = \boxed{\frac{-\pi^2}{8\sqrt{2}}}$$

·) Det. i punti di sing. isolate di $f(z) = \dfrac{e^{2/z}}{z+1}$, classificarli e residui.

Punti $z=0$ e $z=-1$.

Per $z=0$ Laurent $z \in$ (disco)

$$f(z) = e^{2/z}\,\frac{1}{1+z} = \sum_{n=0}^{\infty} \frac{2^n}{n!}\,\frac{1}{z^n} \cdot \sum_{m=0}^{\infty} (-1)^m z^m$$

$$= \sum_{n\in\mathbb{Z}} a_n z^n \sum_{m\in\mathbb{Z}} b_m z^m = \sum_{k\in\mathbb{Z}} \Big(\sum_{\substack{n,m\in\mathbb{Z}\\ n+m=k}} a_n b_m \Big) \cdot z^k$$

$$\sum_{\substack{n,m\in\mathbb{Z}\\ n+m=k}} a_n b_m = \sum_{\substack{n\ge 0\\ m\ge 0\\ m=k+n}} \frac{2^n}{n!} (-1)^m .$$

se $k \ge 0$ $\displaystyle\sum_{n\ge 0} \frac{2^n}{n!} (-1)^{k+n} = (-1)^k \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} = (-1)^k e^{-2}$

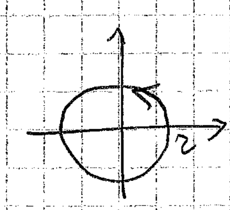
se $k < 0$ $\displaystyle\sum_{n=-k}^{\infty} \frac{2^n}{n!} (-1)^{k+n} = (-1)^k \sum_{n=-k}^{\infty} \frac{(-2)^n}{n!} = (-1)^k \Big(e^{-2} - \sum_{n=0}^{-k-1} \frac{(-2)^n}{n!} \Big)$

Tutti $\neq 0$ $^{(*)}$ $\Rightarrow$ $z=0$ è sing. essenziale.

$$\operatorname{Res}_{z=0} f(z) = -\Big(e^{-2} - \sum_{n=0}^{0} \frac{(-2)^n}{n!} \Big) = 1 - e^{-2}$$

(*) (e^2 non è razionale).

In $z=-1$ $\displaystyle\operatorname{Res}_{z=-1} f(z) = \lim_{z\to -1} (z+1)\,\frac{e^{-2/z}}{z+1} = e^{-2}$ ▨

H Calcolare $\displaystyle\int_\gamma \frac{\sin(\pi/z)}{z^2-1}\,dz$ $\gamma =$ 

con $r > 0$.

$\dfrac{\sin(\pi/z)}{z^2-1}$ Singolarità: $z=0$, $z=1$ e $z=-1$

NA $z = 1$ è eliminabile perché $\lim_{z\to 1} f(z) = \frac{\pi}{2}$

lo stesso in $z = -1$.

in $z = 0$: $f(z) = \sin\left(\frac{\pi}{z}\right) \cdot \frac{1}{z^2-1} = -\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} \frac{1}{z^{2n+1}} \cdot \sum_{m=0}^{\infty} z^{2m}$

$$= \sum_{k\in\mathbb{Z}} \left(-\sum_{\substack{m,n\geq 0 \\ 2m-2n-1=k}} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} \right) z^k$$

se $k = -1$ $\quad -\sum_{\substack{m,n\geq 0 \\ 2m-2n-1=-1}} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = -\sin\pi = 0$

Quindi $\int_\gamma \frac{\sin(\pi/z)}{z^2-1}\, dz = 2\pi i \cdot \operatorname{Res}_{z=0} f(z) = 2\pi i \cdot 0 = 0$

alternativa : $\int_\gamma (\ )\, dz = 2\pi i \operatorname{Res}_{z=0} f(z)$, quindi non dip. da z

ma $\left| \int_{\gamma_z} \frac{\sin(\pi/z)}{z^2-1}\, dz \right| \leq \int_0^{2\pi} \frac{\left| e^{i\frac{\pi}{z} e^{i\theta}} - e^{-i\frac{\pi}{z} e^{i\theta}} \right|}{z^2-1}\, z\, d\theta$

$$\leq \frac{z}{z^2-1} \int_0^{2\pi} \left(e^{\frac{\pi}{z}\sin\theta} + e^{-\frac{\pi}{z}\sin\theta} \right) d\theta$$

$$\leq \frac{z}{z^2-1} \cdot 2e^{\pi/z} \cdot 2\pi \longrightarrow 0 \text{ per } z\to\infty$$, quindi tesi $\blacksquare$

H

$f(z) = \frac{\sin\left(\frac{1}{4z}\right)}{z\left(e^{i/z}-1\right)}$ Det. singolarità isolate e residui.

Den

$z = 0$, e $e^{i/z} = 1$ $\quad \frac{i}{z} = 2\pi i k \quad z = \frac{1}{2\pi k} \quad k\in\mathbb{Z},\ k\neq 0.$

Num $\quad \frac{1}{4z} = l\pi$ ovvero $z = \frac{1}{4\pi l}$, $l\neq 0$.

Quindi : punti $\frac{1}{4\pi l}$, $l\in\mathbb{Z}$, $l\neq 0$ sono singolarità eliminabili

$\Rightarrow$ $\underline{z=0}$ $\underline{\text{Non}}$ è isolata.

$\Rightarrow$ e $z = \dfrac{1}{2\pi(2l+1)}$, $l \in \mathbb{Z}$, sono $\underline{\text{poli}}$ $\underline{\text{semplici}}$

$$\operatorname*{Res}_{z=\frac{1}{2\pi(2l+1)}} f(z) = \lim_{z\to\frac{1}{2\pi(2l+1)}} \frac{\sin\left(\frac{1}{4z}\right)}{z\left(e^{i/z}-1\right)} \cdot \left(z - \frac{1}{2\pi(2l+1)}\right)$$

$$= 2\pi(2l+1)\sin\left(\frac{\pi}{2}(2l+1)\right)\cdot \lim_{z\to z_0} \frac{z-z_0}{e^{i/z}-1}$$

$$= (-1)^l\, 2\pi(2l+1)\cdot \frac{1}{-\frac{i}{z^2}e^{i/z}} = (-1)^l\, 2\pi(2l+1)\, i\, \frac{1}{(2\pi)^2(2l+1)^2}$$

$$= \frac{(-1)^l i}{2\pi(2l+1)}$$

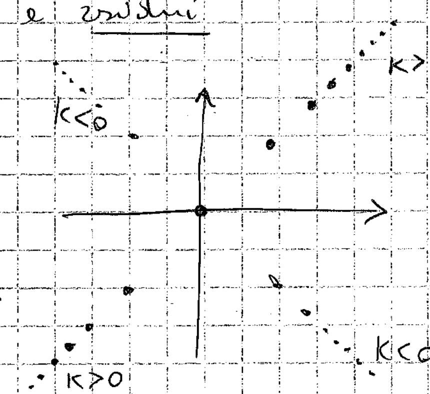
$\blacksquare$

H

$f(z) = \dfrac{1}{e^{z^2}-1}$ sing. isolate e residui

$z^2 = 2k\pi i \qquad k \in \mathbb{Z}$

$\Rightarrow$ Singolarità: $\{0\} \cup \left\{\sqrt{2k\pi}\, e^{\frac{il\pi}{4}},\ k \in \mathbb{N},\ k>0 \text{ e } l=1,3,5,7\right\}$



$\boxed{z=0}$ è polo $\underline{\text{doppio}}$

$\operatorname*{Res}_{z=0} f(z) = 0$ (è pari)

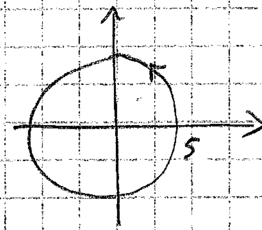
$\underline{\text{Altri}}$ sono poli $\underline{\text{semplici}}$, e $\operatorname*{Res}_{z=z_{k,l}} f(z) = \lim_{z\to z_{k,l}} \dfrac{z-z_{k,l}}{e^{z^2}-1}$

$$= \frac{1}{2z_{k,l}\, e^{z_{k,l}^2}} = \frac{1}{2z_{k,l}}.$$

$\blacksquare$

11

$$F(z) = \oint_{|w|=5} \frac{\sin w}{w^2 - 2wz}\, dw$$



$$F(z) = \int_{\gamma} \left(\frac{\sin w}{w}\right) \cdot \frac{1}{w-2z}\, dw \qquad \begin{cases} \text{se } |2z| < 5 & \dfrac{\sin(2z)}{2z} \\ \text{se } |2z| > 5 & 0 \end{cases}$$

se $|z| = 5/2$ Non definita.

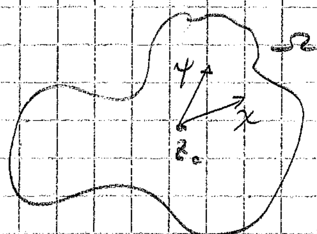
$F(z) \in \mathcal{H}(\mathbb{C} \setminus \bigcirc)$. (Due componenti connesse).

FINE I PARTE

II PARTE

(⇒) $f: \Omega \subseteq \mathbb{C} \to \mathbb{C}$, Ω aperto, f olomorfa.

Se $f'(z) \neq 0$ in Ω, allora f è conforme in Ω (cioè conserva gli angoli orientati)



$$f(z_0 + t\chi) = f(z_0) + f'(z_0)\, t\chi + o(t) \qquad t \in \mathbb{R}$$

$$f(z_0 + t\psi) = f(z_0) + f'(z_0)\, t\psi + o(t)$$

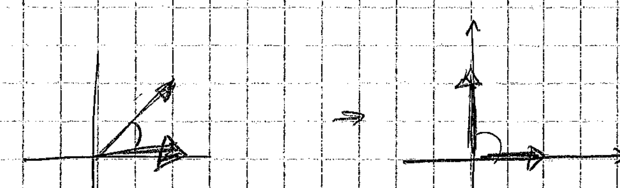
$$\Rightarrow \quad f(z_0 + t\chi) - f(z_0 + t\psi) = t\, f'(z_0)\,(\chi - \psi) + o(t)$$

ora $$t\, f'(z_0) = |t\, f'(z_0)| \frac{f'(z_0)}{|f'(z_0)|} \cdot (\chi - \psi) + o(t)$$

omotetia $\quad$ rotazione ($f'(z_0) \neq 0$) $\quad$ angolo

Fallisce se $f'(z) = 0$.

Ad esempio $f(z) = z^2$ $\to$

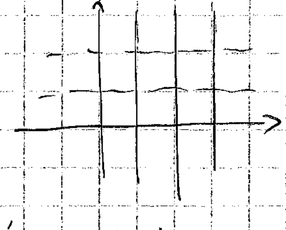


Conseguenza: data f con $f'(z) \neq 0$, la famiglia di curve $\{\operatorname{Re} f(z) = c\}_{c \in \mathbb{R}}$ e la famiglia $\{\operatorname{Im} f(z) = c'\}_{c' \in \mathbb{R}}$ sono $\perp$

(perché $\nabla \operatorname{Re} f$ e $\nabla \operatorname{Im} f$ sono $\perp$)

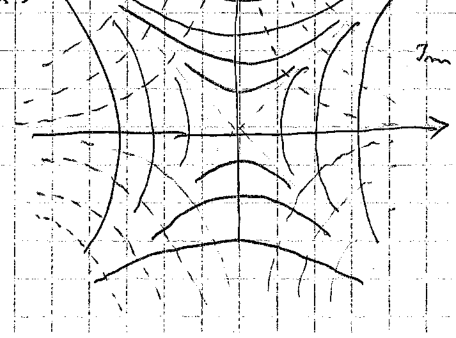
esempio $z = x + iy$

$f(z) = z \Rightarrow x$ e y

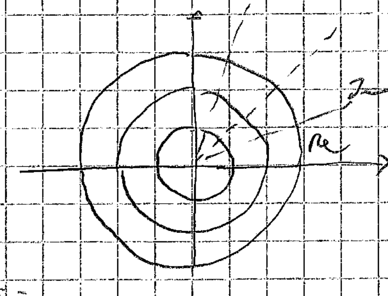


(Banale)

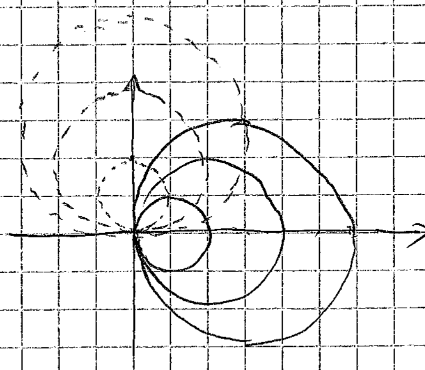
$f(z) = z^2 \Rightarrow x^2 - y^2$ e xy



$f(z) = \lg z \qquad \lg|z| \qquad$ angolo

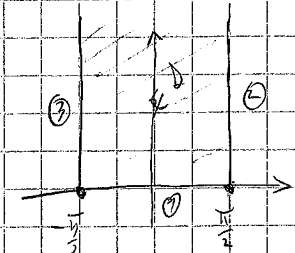


$f(z) = \frac{1}{z} \qquad \frac{x}{x^2+y^2} \qquad \frac{-y}{x^2+y^2}$



•) $f(z) = \sin z \qquad z = x+iy$

$D = \{ |x| \leq \pi/2,\ y \geq 0 \} \qquad f(D) = ?$

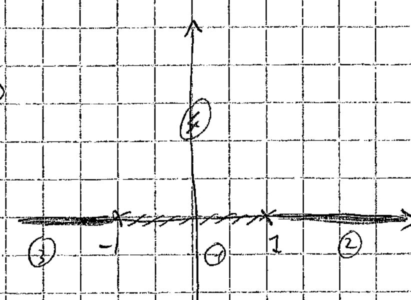


$f((-\frac{\pi}{2}, \frac{\pi}{2})) = (-1, 1)$

$f(\frac{\pi}{2} + i(0,+\infty)) = (1,+\infty)$

$f(\frac{\pi}{2} + iy) = \operatorname{ch} y$

$f(-\frac{\pi}{2} + iy) = -\operatorname{ch} y$



$f(i(0,+\infty)) = i(0,+\infty)$

$(f(iy) = i \operatorname{sh} y)$

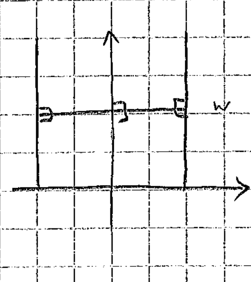
Inoltre D connesso $\Rightarrow f(D)$ connesso,

$\mathring{D}$ open $\Rightarrow f(\mathring{D})$ open (Mappa aperta)

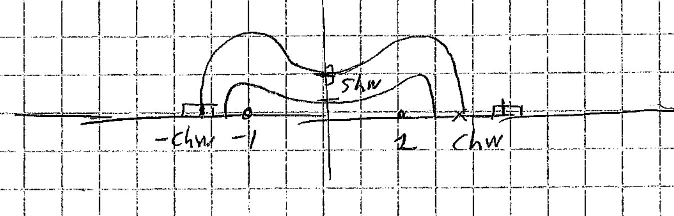
f Biunivoca su $\mathring{D} \Rightarrow f^{-1}$ è olomorfa e manda il bordo di

$f(\mathring{D})$ in $\partial D \Rightarrow$ bordo di $f(\mathring{D}) =$ ——— $\Rightarrow f(D) =$ 

Oss.



$\Rightarrow$



•) Def. $f : \mathbb{C} \to \mathbb{C}$ è AUTOMORFISMO quando

f è biunivoca e intera.

Th. $f \in Aut(\mathbb{C}) \Rightarrow f(z) = az + b \quad (a, b \in \mathbb{C})$

Proof f è intera, $\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n z^n$ con raggio ∞.

$\Rightarrow f(\frac{1}{z}) = \sum_{n=0}^{\infty} a_n z^{-n}$ è lo sviluppo di Laurent di f a ∞.

se ∞ è singolarità essenziale per f (ovvero 0 è sing. ess. per $f(\frac{1}{z})$)

allora $f(|z| > c)$ è denso in $\mathbb{C}$ (Th. Casorati + Weierstrass)

quindi $f(\underset{\text{Aperto}}{|z| < 1}) \cap f(\underbrace{|z| > 1}_{\text{denso}})$ è $\neq \emptyset \Rightarrow \exists\, w_0 \in \mathbb{C}$, con

$\Rightarrow f(\,)$ aperto

ottengo due soluzioni: $f(z) = w_0$: uno $|z_0| < 1$ e $|z_1| > 1$. $\Rightarrow$ imp.

Allora ∞ è polo $\Rightarrow$ f è polinomio ma è iniettivo $\Rightarrow$ f è lineare ∎

•) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to f_M(z) = \dfrac{az+b}{cz+d}$ è Morfismo: $f_{MM'} = f_M \circ f_{M'}$

invertibile $\Leftrightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$. la $f_M : \mathbb{C}^* \to \mathbb{C}^*$

$f_M(\infty) = \frac{a}{c}$ (se $c \neq 0$, ∞ altrimenti).

oss $f_{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) M} = f_M \Rightarrow$ solo tre "parametri liberi" per dire che

f_M è rappr. dal $GL(2, \mathbb{C}) / \langle \lambda I \rangle = SL(2, \mathbb{C})$.

Dati tre punti distinti di $\mathbb{C}^*$ (α, β, γ) e tre punti distinti $(\alpha', \beta', \gamma') \in \mathbb{C}^*$

$\exists!\ f_M$ con $\alpha \to \alpha'$, $\beta \to \beta'$, $\gamma \to \gamma'$

Dim oss. $\alpha' = 0$, $\beta' = \infty$, $\gamma' = 1$: $f(z) = \dfrac{z - \alpha}{z - \beta} \cdot \dfrac{\gamma - \beta}{\gamma - \alpha}$

manda $\alpha \to \alpha' = 0$, $\beta \to \beta' = \infty$, $\gamma \to \gamma' = 1$

Prendo f $\alpha, \beta, \gamma \to (0, \infty, 1)$ $\tilde{f}$ $\alpha', \beta', \gamma' \to (0, \infty, 1)$ allora $\tilde{f}^{-1} \circ f$ è la soluzione

f è unica se f e $\tilde f$: $(\alpha,\beta,\gamma) \to (\alpha',\beta',\gamma')$

allora $f\circ \tilde f^{-1}$ fissano α, β, e γ.

Ma allora $f\circ\tilde f^{-1} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ $\quad (f\circ f^{-1})(\alpha)=\alpha$

$\frac{a\alpha+b}{c\alpha+d}=\alpha \Rightarrow c\alpha^2+(d-a)\alpha+b=0$ lo stesso per β e γ

$\Rightarrow$ l'eq. (II grado) ha tre sol $\Rightarrow$ eq. banale $\Rightarrow c=b=0$ e $a=d$. ∎

Es $(2,i,-2) \to (1,3,i)$

$2, i, -2 \to 0, \infty, 1 \qquad f_1: \frac{z-2}{z-i}\cdot\frac{-2-i}{-2-2} = \frac{2+i}{4}\cdot\frac{z-2}{z-i}$

$1, 3, i \to 0, \infty, 1 \qquad f_2: \frac{z-1}{z-3}\cdot\frac{i-3}{i-1}$

$(2,i,-2) \to (1,3,i)$

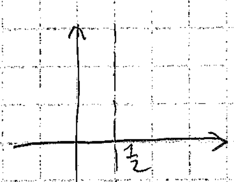
$$f_2^{-1} f_1 \quad \left[\underbrace{\begin{pmatrix}\frac{i-3}{i-1}&0\\0&1\end{pmatrix}\begin{pmatrix}1&-1\\1&-3\end{pmatrix}}\right]^{-1}\underbrace{\begin{pmatrix}\frac{2+i}{4}&0\\0&1\end{pmatrix}\begin{pmatrix}1&-2\\1&-i\end{pmatrix}} = \begin{pmatrix}-\frac18 & -\frac34+\frac i2\\ -\frac38 & -\frac14+\frac i2\end{pmatrix}$$

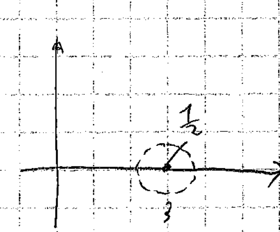
$$f_2^{-1}f_1(z) = \frac{-z-6+4i}{-3z-2+4i}$$

⊢ Le Moebius mandano {rette ∪ circ} → {rette ∪ circ}

Data retta $r: x = \frac12$ e $C: 4(x-3)^2+4y^2 = 1$

trovare $f: r \to C$





1° modo: fisso f imponendo l'immagine di tre punti:

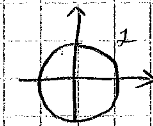
$f(\infty) = 3+\frac12 \quad \rightsquigarrow \quad f(z) = 3+\frac12 + \frac{b}{z+d}$

$f(\frac12+i) = 3+i/2 \qquad \begin{cases} 3+\frac12 + \frac{b}{\frac12+i+d} = 3+i/2 \\ 3+\frac12 + \frac{b}{\frac12-i+d} = 3-i/2 \end{cases}$

$f(\frac12-i) = 3-i/2$

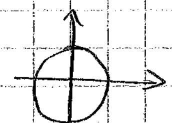
$$\begin{cases} 2b+(1-i)d = -\frac32-\frac i2 \\ 2b+(1+i)d = -\frac32+\frac i2 \end{cases} \qquad \begin{cases} b=-1 \\ d = 1/2 \end{cases} \qquad f(z) = \frac{14z+3}{4z+2}$$

<u>Alternativa</u>

$\phi(z) = \frac{1+z}{1-z}$ manda $i\mathbb{R} \to$ 

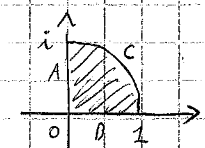
quindi rette $z \to e^{i\theta}z$ così $e^{i\theta}z$ è verticale.

$\xrightarrow[\text{traslazione}]{}$ $e^{i\theta}z + a$ così $\underline{i\mathbb{R}}$

$\to$ Applico $\phi(e^{i\theta}z+a)$ è 

$\to$ Omotetia di raggio di cerchio $\lambda\,\phi(e^{i\theta}z+a)$

$\to$ Traslo nel centro c: $\lambda\,\phi(e^{i\theta}z+a)+c$

•) Dato $f(z) = \frac{z+\frac{1}{2}}{z-i}$ $D =$ 

$f(D) = ?$

$f(0) = \frac{i}{2}$

$f(i) = \infty$

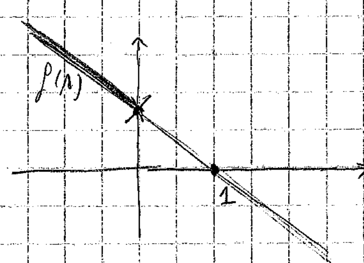
$f(1) = \frac{3/2}{1-i} = \frac{3}{4}(1+i)$

$f(A) = ?$

$f(i) = \infty$

$f(0) = \frac{i}{2}$

$f(\infty) = 1$

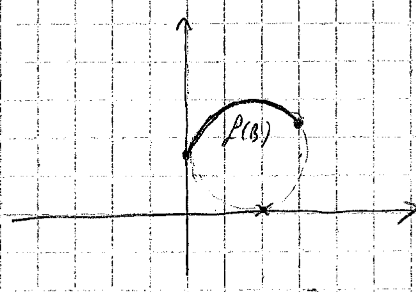


$f(B) = ?$

$f(0) = i/2$

$f(1) = \frac{3}{4}(i+1)$

$f(\infty) = 1$ $\Rightarrow$ arco

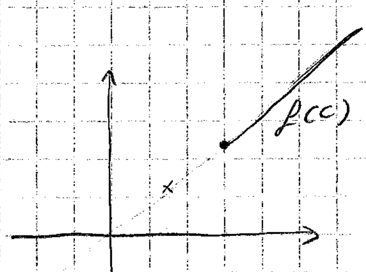


$f(C) = ?$

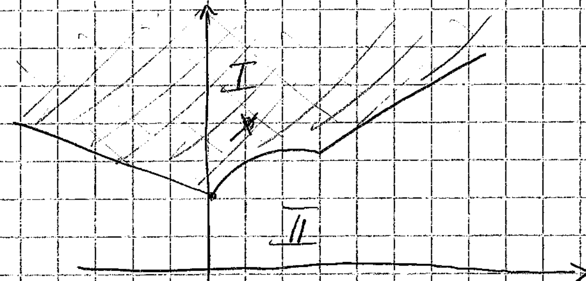
$f(i) = \infty$ $\Rightarrow$ retta

$f(1) = \frac{3}{4}(i+1)$

$f(-i) = \frac{-i+\frac{1}{2}}{-2i} = \frac{1}{2} + \frac{i}{2}$



$\Rightarrow f\left(\ \right) =$



quindi $f(\ ) = I \cup II$

Ma $f\left(\frac{1}{2}+\frac{i}{2}\right) = \frac{2+i}{1-i} = \frac{1+3i}{2}$

$\cap$

$\vdash$ Formula per $\sin(\pi z)$.

Obiettivo: Verificare che $\dfrac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty}\left(1-\dfrac{z^2}{n^2}\right) \quad \forall z \in \mathbb{C}$.

Lemma i) $\pi \cot(\pi z) = \dfrac{1}{z} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(\dfrac{1}{z-n}+\dfrac{1}{n}\right) = \dfrac{1}{z} + \sum_{n=1}^{\infty} \dfrac{2z}{z^2-n^2} \quad \forall z \in \mathbb{C}$

ii) $\dfrac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \dfrac{1}{(z-n)^2} \quad \forall z \in \mathbb{C}$.

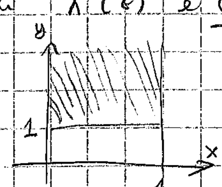
Dim Sia $h(z) = \pi \cot(\pi z) - \left[\dfrac{1}{z} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(\dfrac{1}{z-n}+\dfrac{1}{n}\right)\right] =: h_1(z) - h_2(z)$

h è olo su $\mathbb{C}\setminus\mathbb{Z}$. i punti $z\in\mathbb{Z}$ sono poli semplici per $h_1(z)$ e $h_2(z)$ ma hanno residuo uguale $\Rightarrow$ sono sing. el. per h $\Rightarrow$ h è intera.

$h'(z) = -\dfrac{\pi^2}{\sin^2(\pi z)} + \sum_{n\in\mathbb{Z}} \dfrac{1}{(z-n)^2}$ (facile)

$\Rightarrow h(z)$ è 1-periodica.

Mostro che $h'(z)$ è costante: per Liouville, basta mostrare che è limitata.



- se $z = x+iy$ e $0\leq x\leq 1$, $|y|\leq 2$ ovv. limitata.

$\rightarrow$ se $0\leq x\leq 1$ e $|y|>2$, allora

$$\frac{1}{|z-n|^2} = \frac{1}{(x-n)^2+y^2} \leq \frac{1}{(|n|-1)^2+4} \quad \text{quindi}$$

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z-n|^2} \leq \sum_{n\in\mathbb{Z}} \frac{1}{(|n|-1)^2+4} < +\infty \quad \text{per } 0\leq x\leq 1,\ |y|\geq 2.$$

$\rightarrow$ $\left|\dfrac{1}{\sin^2(\pi(x+iy))}\right| = \dfrac{4}{|e^{i\pi x-\pi y} - e^{-i\pi x+\pi y}|^2} \leq \dfrac{4}{|e^{\pi y}-e^{-\pi y}|^2} = \dfrac{1}{|\sinh \pi y|^2}$ limitata in $|y|\geq 2$.

EULER

sappiamo che $\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$

47
B'2

$\Rightarrow \quad \pi z \cot(\pi z) - 1 = \sum_{n=1}^{\infty} \frac{2z^2}{z^2-n^2}$ (*)

$\boxed{\text{Ma}}$: $\sum_{n=1}^{\infty} \frac{z^2}{z^2-n^2} = -\sum_{n=1}^{\infty} \frac{z^2}{n^2} \frac{1}{1-\frac{z^2}{n^2}} = -\sum_{k=1}^{\infty} z^{2k} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = -\sum_{k=1}^{\infty} \zeta(2k)\, z^{2k}$ (**)

e posto $\frac{u}{e^u-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} u^n$, si ha

$$u \cot u - 1 = iu \frac{e^{2iu}+1}{e^{2iu}-1} - 1 = iu \left[1 + \frac{2}{e^{2iu}-1}\right] - 1 = \frac{2iu}{e^{2iu}-1} + iu - 1$$

$$= \sum_{n=0}^{\infty} \frac{B_n (2i)^n u^n}{n!} + iu - 1 = \sum_{n=2}^{\infty} \frac{B_n (2i)^n}{n!} u^n = \sum_{k=1}^{\infty} \frac{B_{2k}(-1)^k 2^{2k}}{(2k)!} u^{2k}$$ (***)

$B_0 = 1$
$B_1 = -\frac{1}{2}$

$B_n = 0$ se n è dispari e ≥ 3

quindi, da (*), (**) e (***) segue

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} 2^{2k} (-1)^k \pi^{2k} z^{2k} = -2\sum_{k=1}^{\infty} \zeta(2k)\, z^{2k}$$

ovvero $\quad \zeta(2k) = (-1)^{k-1} \frac{B_{2k}}{2} \cdot \frac{(2\pi)^{2k}}{(2k)!} \qquad \forall\, k$

$\zeta(2) = \pi^2/6, \quad \zeta(4) = \frac{\pi^4}{90}, \ \ldots$

$\Rightarrow$ h' è limitato $\Rightarrow$ h' è costante.

·) $h'(z) = 0$ infatti $|h'(iy)| \leq \frac{\pi^2}{Sh(\pi y)^2} + \sum_{n\in\mathbb{Z}} \frac{1}{(|n|-1)^2+y^2}$

$$\leq \frac{\pi^2}{(Sh\,\pi y)^2} + \underset{n=0}{\frac{1}{1+y^2}} + \underset{n=\pm 1}{\frac{2}{y^2}} + 2\int_1^\infty \frac{dm}{m^2+y^2}$$

$$= \frac{\pi^2}{(Sh\,\pi y)^2} + O\left(\frac{1}{y^2}\right) + O\left(\frac{1}{y}\right) \underset{y\to\infty}{\longrightarrow} 0.$$

$\Rightarrow$ $\underline{\underline{h'(z) = 0}}$

questo dimostra (ii) e ciò implica $\underline{h(z)}$ è $\underline{\text{costante}}$.

$\underline{\text{ma}}$ $h(z)$ è $\underline{\text{dispari}}$ (facile: $\cot z$ lo è, $\frac{1}{z}$ lo è, e

$$\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(\frac{1}{-z-n} + \frac{1}{n}\right) = -\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(\frac{1}{z+n} + \frac{1}{-n}\right) \underset{n\to -n}{=} -\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left(\frac{1}{z-n} + \frac{1}{n}\right) \Bigg)$$

quindi $\underline{h(z) = 0}$ $\Rightarrow$ (i) è vera. ▨

$\underline{\text{Dim}}$ formula per $\underline{\text{seno}}$. $\frac{\sin(\pi z)}{\pi z}$ e $\prod_{n=1}^\infty \left(1-\frac{z^2}{n^2}\right)$ hanno zeri negli stessi punti e stessa $\underline{\text{molteplicità}}$. $\Rightarrow$ il quoziente è $\underline{\text{intero}}$ e $\neq 0$. $\Rightarrow$ $\lg\left(\frac{\sin(\pi z)}{\pi z} / \prod\right)$ è intero. $\Rightarrow$ la sua derivata $\underline{\text{esiste}}$. Ma parte è $\pi \cot z - \frac{1}{z} + \sum_{n=1}^\infty \frac{2z}{n^2-z^2}$ ma (i) dice che questo è 0 $\Rightarrow$

$\exists$ c costante t.c. $\frac{\sin(\pi z)}{\pi z} = c \cdot \prod_{n=1}^\infty \left(1-\frac{z^2}{n^2}\right)$. prendo $z=0$ $\to$ $c=1$ ▨

(*) $\underline{\text{Vd.}}$ Calcolo di Eulero su $\zeta(2), \zeta(4), \ldots$ Su (47 Bis)

$\underline{\text{La } \Gamma(z)}$

→ $\Gamma(z) = \int_0^\infty e^{-t} t^z \frac{dt}{t}$ $\mathrm{Re}\, z > 0$. è $\underline{\text{olo}}$ per $\mathrm{Re}\, z > 0$.

→ Una integrazione per parti dà: $\Gamma(z+1) = \int_0^\infty e^{-t} t^z dt = -e^{-t}t^z\Big|_0^\infty + \int_0^\infty e^{-t} z t^z \frac{dt}{t} = z\Gamma(z)$

ovvero $\Gamma(z+1) = z\Gamma(z)$ se $\mathrm{Re}\, z > 0$.

→ $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$ e $\Gamma(n+1) = n\Gamma(n)$ $\Rightarrow$ $\Gamma(n) = (n-1)!$ $\forall n \geq 1$.

-) $\rightsquigarrow$ $\Gamma(z) = \frac{1}{z}\Gamma(z+1)$ che dà prolungamento analitico di Γ a $\mathrm{Re}\, z > -1$, con polo semplice in $z=0$, residuo $=1$.

·) $\Gamma(z) = \frac{1}{z(z+1)}\Gamma(z+2)$ prolungamento di Γ a $\mathrm{Re}\, z > -2$, con polo semplice in $z=-1$, residuo -1.

Iterando: $\Gamma(z) = \frac{\Gamma(z+m+1)}{z(z+1)\cdots(z+m)}$ prolungamento di Γ a $\mathrm{Re}\, z > -m$, con polo semplice in $z=-m$, residuo $\frac{(-1)^m}{m!}$

<u>Quindi</u>: Γ è meromorfa su $\mathbb{C}$, unici poli sono semplici in $z=-m$ $(m \in \mathbb{N})$ e residuo è $\frac{(-1)^m}{m!}$.

$\vdash$ Se $\mathrm{Re}\, z > 0$, $\Gamma(z) = \int_0^\infty e^{-t} t^z \frac{dt}{t}$

ma $e^{-t} = \lim_{n\to\infty} \left(1-\frac{t}{n}\right)^n \chi_{[0,n]}(t)$ Inoltre $\left(1-\frac{t}{n}\right)^n = e^{n\log\left(1-\frac{t}{n}\right)} \le e^{-t}$ su $t \in [0,n]$

$\Rightarrow$ per convergenza dominata:

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \frac{dt}{t} = \lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt \qquad (*)$$

<u>ma</u> $\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt = \frac{n!\, n^z}{z(z+1)\cdots(z+n)}$

(Integrazioni successive per <u>parti</u>)

Quindi $$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)}$$ <u>$\mathrm{Re}\, z > 0$</u> (In realtà vale $\forall z$, dalla eq. funzionale).

Quindi $$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} z\, n^{-z} \prod_{j=1}^n \left(1+\frac{z}{j}\right) = \lim_{n\to\infty} z\, e^{z\left(\sum_{j=1}^n \frac{1}{j} - \log n\right)} \prod_{j=1}^n \left(1+\frac{z}{j}\right) e^{-z/j} \qquad (**)$$

·) $\exists\, \gamma$ $\sum_{j=1}^n \frac{1}{j} - \log n \to \gamma$

Infatti se $u_n = \sum_{j=1}^n \frac{1}{j} - \log n$ allora $u_{n+1} - u_n = \frac{1}{n+1} - \log\left(\frac{n+1}{n}\right) \le \frac{1}{n+1} - \frac{1}{n} < 0$.

e da $\frac{1}{k+1} \le \int_k^{k+1} \frac{du}{u} \le \frac{1}{k}$ segue $0 \le u_n \le 1$.

quindi u_n <u>converge</u>.

quindi passando al limite in (**) si ha

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}$$

→ la conv. è assoluta e unif. sui compatti ⇒ RHS è olomorfa su $\mathbb{C}$ ⇒ l'identità vale su $\mathbb{C}$

⇒ $\Gamma(z)$ non ha zeri.

•) Inoltre

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{1}{\Gamma(z)\Gamma(-z)(-z)} = \frac{z}{(-z)} \cdot e^{\gamma z} \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n} \cdot (-z)\, e^{-\gamma z}\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n}$$

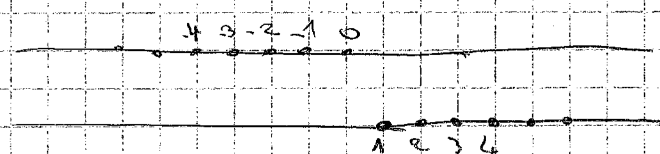
$$= z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \frac{1}{\pi}\sin(\pi z)$$

quindi

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \qquad \left(\text{e } \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\right)$$

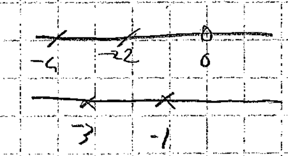
$\Gamma(z)$

$\Gamma(1-z)$



$$\Rightarrow \frac{1}{\sin(\pi z)}$$

•) Duplicazione: $\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) =$



$= \Gamma(z) \cdot e^{\text{pol } 1^\circ \text{ grado}}$

$$\Gamma\left(\frac{z}{2}\right)^{-1}\Gamma\left(\frac{z+1}{2}\right)^{-1} = \frac{z}{2}\cdot e^{\gamma z/2}\,\frac{z+1}{2}\, e^{\gamma\frac{z+1}{2}} \prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)e^{-\frac{z}{2n}} \prod_{n=1}^{\infty}\left(1+\frac{z+1}{2n}\right)e^{-\frac{z+1}{2n}}$$

$$= z\, e^{\gamma z}\,\frac{e^{\gamma/2}}{4}(1+z) \cdot \prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)e^{-z/2n} \cdot \prod_{n=1}^{\infty}\left(1+\frac{z}{2n+1}\right)\frac{2n+1}{2n}\, e^{-\frac{z+1}{2n}}$$

$$= z\, e^{\gamma z} \quad 〃 \quad 〃 \quad \cdot \quad 〃 \quad \cdot \prod_{n=1}^{\infty}\left(1+\frac{z}{2n+1}\right)\cdot e^{-\frac{z}{2n+1}} \cdot \left(1+\frac{1}{2n}\right)\cdot e^{-\frac{z+1}{2n}+\frac{z}{2n+1}}$$

$$= z\, e^{\gamma z}\,\frac{e^{\gamma/2}}{4} \cdot \prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)e^{-z/2n} \cdot \prod_{\substack{n=1\\ \text{odd}}}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n} \cdot e^{z} \cdot \prod_{n=1}^{\infty}\left(1+\frac{1}{2n}\right)e^{-\frac{1}{2n}} \cdot \underbrace{\prod_{n=1} e^{z\left(\frac{1}{2n+1}-\frac{1}{2n}\right)}}_{e^{z(\log 2-1)}}$$

$$= 2^{z}\,\Gamma(z)^{-1} \cdot \frac{e^{\gamma/2}}{4} \cdot \prod_{n=1}^{\infty}\left(1+\frac{1}{2n}\right)e^{-\frac{1}{2n}}$$

quindi $\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) = 2^{-z}\,\Gamma(z) \cdot c$ con $c \in \mathbb{C}$, $c \neq 0$.

per trovare c: prendo $z \to 0$

$$\frac{2}{z}\cdot\Gamma(1/2) = 2^{o(1)}\cdot c\cdot\frac{1}{z} \Rightarrow c = 2\Gamma(1/2) = 2\sqrt{\pi}$$

Quindi $$\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) = 2^{1-z}\sqrt{\pi}\,\Gamma(z)$$

In generale, $\forall$ m intero >0,

$$\prod_{j=0}^{m-1}\Gamma\left(\frac{z+j}{m}\right) = (2\pi)^{\frac{m-1}{2}}\cdot m^{\frac{1}{2}-z}\,\Gamma(z)$$ ■

⊢ Stirling

Strumento: Euler-Maclaurin formula

Se $f \in C^1$ basta. Integrazione per parti

$$\int_0^1 f(x)\,dx = \left(x-\tfrac12\right)f(x)\Big|_0^1 - \int_0^1 \left(x-\tfrac12\right)f'(x)\,dx = \frac12 f(0)+\frac12 f(1) - \int_0^1\left(x-\tfrac12\right)f'(x)\,dx$$

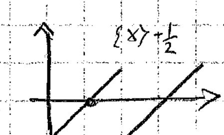
ovvero

$$\frac12 f(0)+\frac12 f(1) = \int_0^1 f(x)\,dx + \int_0^1\left(x-\tfrac12\right)f'(x)\,dx \qquad (*)$$

lo applico a $f_j(x) := f(j+x)$ e sommo su $j=0,\dots,n-1$

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx + \frac12 f(n)+\frac12 f(0) + \int_0^n \left(\{x\}-\tfrac12\right)f'(x)\,dx \qquad (**)$$

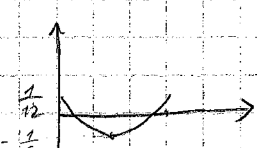
che dà confronto tra Σ e $\int$.



•) Un passo in più: Integro (*) per parti:

come primitiva di $x-\frac12$ prendo $\frac12\left(x-\frac12\right)^2-\frac{1}{24} = \frac12\left(x^2-x+\frac16\right)$

che ha media integrale nulla.



$$\Rightarrow \frac12 f(0)+\frac12 f(1) = \int_0^1 f(x) + \frac{1}{12}f'(1) - \frac{1}{12}f'(0) + \frac12\int_0^1 \left(x^2-x+\tfrac16\right)f''(x)\,dx$$

Quindi $$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx + \frac12\left(f(n)+f(0)\right) + \frac{1}{12}\left(f'(n)-f'(0)\right) + \frac12\int_0^n \underbrace{\left(\{x\}^2-\{x\}+\tfrac16\right)}_{B_2(\{x\})} f''(x)\,dx$$

---

ovviamente il procedimento può essere iterato ad ogni ordine, ma per ora questo basta.

Sappiamo che $\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)}$ $\quad z\in\mathbb{C}\setminus\mathbb{Z}$

(la dim è per $\operatorname{Re} z>0$, ma l'eq. funzionale la estende a $z\in\mathbb{C}$).

La vedo come $\Gamma(z) = \lim_{n\to\infty} \frac{1}{z}\cdot\frac{2}{1+z}\cdot\frac{3}{2+z}\cdots\frac{1+n}{z+n}\cdot\frac{n^z}{1+n}$, quindi

$$\log\Gamma(z) = \lim_{n\to\infty}\Big[z\log n - \log(n+1) + S(1) - S(z)\Big] \quad \text{dove } S(z) = \sum_{k=0}^{n}\log(z+k)$$

Per valutare $S(1)$ e $S(z)$ uso Eulero-Maclaurin:

$$\log\Gamma(z) = \lim_{n\to\infty}\Big[z\log n - \log(1+n) + \int_0^n \log(k+1)\,dk - \int_0^n \log(k+z)\,dk + \frac{1}{2}\big(\log(n+1)+\log(1)\big) - \frac{1}{2}\big(\log(z+n)+\log(z)\big)$$
$$+\frac{1}{12}\left(\frac{1}{n+1}-1\right) - \frac{1}{12}\left(\frac{1}{z+n}-\frac{1}{z}\right) + \frac{1}{2}\int_0^n \frac{B_2(\{k\})}{k^2}\,dk - \frac{1}{2}\int_0^n \frac{B_2(\{k\})}{(k+z)^2}\,dk\Big]$$

ora $\int_1^n \frac{B_2(\{k\})}{k^2}\,dk$ e $\int_1^n \frac{B_2(\{k\})}{(k+z)^2}\,dk$ conv. ass. poiché $|B_2(\{k\})|\le 1/6$

$$= \lim_{n\to\infty}\Big[z\log n - \log(n+1) + \underbrace{(n+1)\log(n+1) - \cancel{(n+1)} + \cancel{1}}_{\to -z} - (n+z)\log(n+z) + \cancel{n+z} + z\log z - \cancel{z}$$
$$+\underbrace{\frac{1}{2}\log(n+1) - \frac{1}{2}\log(z+n)}_{\to 0} - \frac{1}{2}\log(z)\Big] - \frac{1}{12} + \frac{1}{12z} + \frac{1}{2}\int_0^\infty \frac{B_2(\{k\})}{k^2}\,dk - \frac{1}{2}\int_0^\infty \frac{B_2(\{k\})}{(k+z)^2}\,dk$$

$$= \left(z-\tfrac{1}{2}\right)\log z - z + C + \frac{1}{12z} - \frac{1}{2}\int_0^\infty \frac{B_2(\{k\})}{(k+z)^2}\,dk$$

dove $C = -\frac{1}{12} + \frac{1}{2}\int_0^\infty \frac{B_2(\{k\})}{k^2}\,dk$

ma $\left|\int_0^\infty \frac{B_2(\{k\})}{(k+z)^2}\,dk\right| \le \frac{1}{6}\int_0^\infty \frac{dk}{(k+x)^2+y^2} \le \frac{1}{6}\int_0^\infty \frac{dk}{k^2+(x^2+y^2)} = \frac{\pi/2}{6|z|}$

$\boxed{x\ge 0}$

Quindi $\log\Gamma(z) = \left(z-\tfrac{1}{2}\right)\log z - z + C + O\!\left(\frac{1}{|z|}\right)$ se $\operatorname{Re} z\ge 0$.

Per trovare C, osservo: $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \Rightarrow \Gamma(it)(-\cancel{i}t)\Gamma(-it) = \frac{\cancel{(-i)}\,\pi\, 2}{e^{\pi t}-e^{-\pi t}}$

quindi $\log(2\pi) - \log\left(e^{\pi t}-e^{-\pi t}\right) = \log\Gamma(it) + \log t + \log\Gamma(-it)$

$\log(2\pi) - \pi t + o(1) = \left(it-\tfrac{1}{2}\right)\log(it) - \cancel{it} + C + \log t + \left(-it-\tfrac{1}{2}\right)\log(-it) + \cancel{it} + C + o(1)$

$= \underline{\left(it-\tfrac{1}{2}\right)\log t + \log t + \left(-it-\tfrac{1}{2}\right)\log t} + \left(it-\tfrac{1}{2}\right)\log(i) + \left(-it-\tfrac{1}{2}\right)\log(-i) + 2C + o(1)$

$= 2C - \pi t + o(1) \implies C = \frac{1}{2}\log(2\pi)$

Conclusione:

$$\log \Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + O\left(\frac{1}{|z|}\right) \quad \text{se } \underline{\operatorname{Re} z \geq 0}.$$

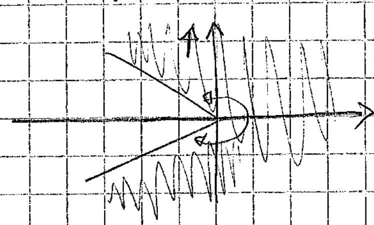
$$\int_0^\infty \frac{dk}{(k+x)^2+y^2} = \int_0^{+\infty} \frac{dk}{k^2+2kx+|z|^2} \underset{\substack{z\neq 0,\\ k \Rightarrow |z| w}}{=} \frac{1}{|z|}\int_0^\infty \frac{dw}{w^2+2\frac{x}{|z|}+1}$$

Chiaramente $-1 \leq \frac{x}{|z|} \leq 1$. Ma, se $x \geq 0$ allora $w^2+\frac{2x}{|z|}+1 \geq w^2+1 \rightsquigarrow$ ok.

se $x<0$, pongo $\frac{x}{|z|} > -1+\varepsilon$, allora $w^2+\frac{2x}{|z|}+1 \geq w^2-2(1-\varepsilon)w+1 \rightsquigarrow$ ok.

$\underline{\text{Unif}}$ in ε

$|z| > \frac{-x}{1-\varepsilon}$



allora $\Gamma(z) = \left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}}\left(1+O\left(\frac{1}{|z|}\right)\right)$

e $n! = \Gamma(n+1) = n\Gamma(n) = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1+O\left(\frac{1}{n}\right)\right)$

$\zeta(z)$ Riemann Zeta function

→ $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ conv. in $\mathrm{Re}\, z > 1$, Unif. sui compatti

⇒ olomorfa qui.

→ Lemma $\mathrm{Re}\, z > 1$ da $\Gamma(z) = \int_0^\infty e^{-t} t^z \frac{dt}{t}$ segue $(t \to nw)$

$$\frac{1}{n^z}\Gamma(z) = \int_0^\infty e^{-nw} w^z \frac{dw}{w}$$ Sommo su n:

$$\Gamma(z)\zeta(z) = \sum_{n=1}^{\infty}\int_0^\infty e^{-nw} w^z \frac{dw}{w} \underset{\text{ASS. conv.}}{=} \int_0^{+\infty}\sum_{n=1}^{\infty} e^{-nw} w^z \frac{dw}{w}$$

$$= \int_0^{\infty} \frac{1}{e^w - 1}\cdot w^z \frac{dw}{w}$$

l'integrale converge "bene" all'∞ (per l'esponenziale) ma in 0 richiede

$2 - \mathrm{Re}\, z < 1$ ⇒ $\mathrm{Re}\, z > 1$, appunto.

Però:

$$\int_0^{+\infty} = \int_0^1 + \int_1^\infty = \int_0^1 \left[\frac{1}{e^w - 1} - \frac{1}{w}\right] w^z \frac{dw}{w} + \int_0^1 w^{z-2}\, dw + \int_1^\infty \frac{w^{z-1}}{e^w - 1}\, dw$$

$$= \frac{1}{z-1} + \int_0^1 \left[\frac{1}{e^w-1} - \frac{1}{w}\right] w^{z-1}\, dw + \int_1^\infty \frac{w^{z-1}}{e^w-1}\, dw \qquad (*)$$

↑ Polo semplice $z=1$ ↑ conv. $\forall\, \mathrm{Re}\, z > 0$ ↑ conv. $\forall\, z \in \mathbb{C}$

⇒ Da $(*)$ $\zeta(z) = \frac{1}{\Gamma(z)}\cdot\left[\quad\right]$ dà prolungamento analitico di $\zeta(z)$ a $\mathrm{Re}\, z > 0$, con unico polo semplice in $z=1$ e residuo 1.

Il processo può essere iterato:

Abbiamo definito i numeri B_n t.c.

$$\frac{w}{e^w-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} w^n = 1 - \frac{1}{2} w + \frac{B_2}{2} w^2 + \dots$$

quindi $$\frac{1}{e^w-1} = \frac{1}{w} - \frac{1}{2} + \frac{B}{2} w + \dots = \sum_{n=0}^{\infty} \frac{B_n}{n!} w^{n-1}$$

Allora $$\frac{1}{e^w-1} - \sum_{n=0}^{L} \frac{B_n}{n!} w^{n-1} = \sum_{n=L+1} \frac{B_n}{n!} w^{n-1}$$

Così: (*) dà

$$\Gamma(z)\zeta(z) = \int_0^1 \left[\frac{1}{e^w-1} - \sum_{n=0}^{L} \frac{B_n}{n!} w^{n-1}\right] w^z \frac{dw}{w} + \sum_{n=0}^{L} \frac{B_n}{n!} \frac{1}{z+n-1} + \int_1^{\infty} \frac{w^{z-1}}{e^w-1} dw$$

da per $\underline{\underline{\mathrm{Re}\, z > -L}}$ — Poli in $z=1,0,-1,\dots -L+1$. — Intera

$\Rightarrow \zeta(z) = \frac{1}{\Gamma(z)}\left[D_*\right]$ Dà prol. analitico in $\mathrm{Re}\, z > -L$, con polo $z=1$ (gli altri poli sono cancellati dai poli di $\Gamma(z)$).

L arbitrario $\Rightarrow$ ζ prolungabile su $\mathbb{C}$, meromorfa, unico polo $\underline{z=1}$.

<u>OSS</u> che $$\sum_{n=0}^{L} \frac{B_n}{n!} \frac{1}{z+n-1} = -\int_1^{\infty} \sum_{n=0}^{L} \frac{B_n}{n!} w^{n-1} \cdot w^{z-1} dw$$

se $\mathrm{Re}\,(z+n-2) < -1$ $\forall n=0,\dots,L$ ovvero $\underline{\underline{\mathrm{Re}\, z < -L+1}}$

Quindi per $\underline{-L < \mathrm{Re}\, z < 1-L}$

$$\Gamma(z)\zeta(z) = \int_0^{\infty} \left[\frac{1}{e^w-1} - \sum_{n=0}^{L} \frac{B_n}{n!} w^{n-1}\right] w^{z-1} dw \qquad (**)$$

Prendo $\underline{L=1}$. ricordo che

$$\pi y \cot(\pi y) - 1 = \sum_{n=1}^{\infty} \frac{2y^2}{y^2-n^2}$$

$$\pi y i \frac{e^{i\pi y} + e^{-i\pi y}}{e^{i\pi y} - e^{-i\pi y}} - 1 = i\pi y \left[\frac{2}{e^{2i\pi y}-1} + 1\right] - 1$$

<u>Quindi</u>

Se pongo $y = \dfrac{w}{2\pi i}$ si ha

$$\frac{w}{e^w - 1} + \frac{w}{2} - 1 = \sum_{n=1}^{\infty} \frac{2w^2}{w^2 + 4\pi^2 n^2}$$

e così $(**)$ dà

$$\Gamma(z)\zeta(z) = 2\int_0^{\infty} \sum_{n=1}^{\infty} \frac{w^z}{w^2 + 4\pi^2 n^2}\, dw \qquad \underline{\underline{-1 < \operatorname{Re} w < 0}}$$

ma se $-1 < x < 0$, $\displaystyle\int_0^{\infty} \sum_{n=1}^{\infty} \frac{w^x}{w^2 + 4\pi^2 n^2}\, dw \underset{\text{Tonelli}}{=} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{w^x}{w^2 + 4\pi^2 n^2}\, dw$

$$\underset{w = 2\pi n \rho}{=} \sum_{n=1}^{\infty} \frac{(2\pi n)^x}{2\pi n} \int_0^{\infty} \frac{\rho^x}{\rho^2+1}\, d\rho = (2\pi)^{x-1} \cdot \underbrace{\zeta(1-x)}_{\text{converge}} \cdot \int_0^{\infty} \frac{\rho^x}{\rho^2+1}\, d\rho < +\infty$$

quindi

$$\Gamma(z)\zeta(z) = 2\sum_{n=1}^{\infty} \int_0^{\infty} \frac{w^z}{w^2+4\pi^2 n^2}\, dw = 2\sum_{n=1}^{\infty} \frac{(2\pi n)^z}{2\pi n} \int_0^{\infty} \frac{\rho^z}{\rho^2+1}\, d\rho$$

$$= 2\cdot(2\pi)^{z-1} \cdot \zeta(1-z) \cdot \underbrace{\int_0^{\infty} \frac{\rho^z}{\rho^2+1}\, d\rho}_{\frac{\pi}{2} \frac{1}{\cos\left(\frac{\pi}{2} z\right)}}$$

$$\Rightarrow \quad \Gamma(z)\zeta(z) = \frac{(2\pi)^z}{2\cos\left(\frac{\pi}{2}z\right)} \zeta(1-z) \qquad \underline{-1 < \operatorname{Re} z < 0}$$

è dimostrata in $-1 < \operatorname{Re} z < 0$, ma ζ e Γ sono meromorfe $\Rightarrow$ vale <u>ovunque</u>.

Inoltre, da Duplicazione e Riflessione, segue:

$$\Gamma\left(\tfrac{z}{2}\right)\Gamma\left(\tfrac{z+1}{2}\right) = 2^{1-z}\sqrt{\pi}\,\Gamma(z) \quad \Rightarrow$$

$$\underline{\pi^{-z/2}\,\Gamma\left(\tfrac{z}{2}\right)\zeta(z)} = \frac{\cancel{2}^{1-z}\sqrt{\pi}\cdot\pi^{-z/2}}{\Gamma\left(\frac{z+1}{2}\right)} \cdot \frac{(2\pi)^z}{\cancel{2}\cos\left(\frac{\pi}{2}z\right)}\,\zeta(1-z) = \underline{\pi^{-\frac{(1-z)}{2}}\,\Gamma\left(\tfrac{1-z}{2}\right)\zeta(1-z)}$$